

TOTAL JOINT ERGODICITY FOR TOTALLY ERGODIC SYSTEMS

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ABSTRACT. Examining multiple ergodic averages whose iterates are integer parts of real valued polynomials for totally ergodic systems, we provide various characterizations of total joint ergodicity, meaning that an average converges to the “expected” limit along every arithmetic progression. In particular, we obtain a complete characterization when the number of iterates is at most two, and disprove a conjecture of the first author. We also improve a result of Frantzikinakis on joint ergodicity of Hardy field functions of at most polynomial growth for totally ergodic systems, which extends a conjecture of Bergelson-Moreira-Richter. Our method is to first use the methodology of Frantzikinakis, which allows one to reduce the systems to rotations on abelian groups without using deep tools from ergodic theory, then develop formulas for integrals of exponential functions over subtori, and finally, compute exponential sums for integer parts of real polynomials.

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1. INTRODUCTION

1.1. The joint ergodicity and total joint ergodicity phenomena. Let (X, \mathcal{B}, μ) be a standard Borel probability space. If $T : X \rightarrow X$ is a measurable transformation which preserves the measure μ (i.e., $\mu(T^{-1}A) = \mu(A)$ for all $A \in \mathcal{B}$), then we say that the quadruple (X, \mathcal{B}, μ, T) is a *measure preserving system* (or just a *system*). We will assume in what follows that the transformation T is invertible. We are interested in the L^2 -limiting behavior, as $N \rightarrow \infty$, of multiple ergodic averages of the form

$$(1) \quad \frac{1}{N} \sum_{n=1}^N T^{a_1(n)} f_1 \cdots T^{a_\ell(n)} f_\ell,$$

where the $(a_i(n))_n$, $1 \leq i \leq \ell$, are suitable integer valued sequences, and the f_i 's are arbitrary bounded functions; for a positive integer n , and a bounded function f , T^n denotes the composition $T \circ \cdots \circ T$ of n copies of T , and $Tf(x) := f(Tx)$, $x \in X$.

In particular, we are interested in the case where the limit is the ‘‘expected’’ one.

Definition. For a collection of sequences $a_1, \dots, a_\ell : \mathbb{N} \rightarrow \mathbb{Z}$, and a system (X, \mathcal{B}, μ, T) , we say that $(a_1(n))_n, \dots, (a_\ell(n))_n$ are

- *jointly ergodic for (X, \mathcal{B}, μ, T)* , if for all functions $f_1, \dots, f_\ell \in L^\infty(\mu)$ we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^{a_1(n)} f_1 \cdots T^{a_\ell(n)} f_\ell = \int_X f_1 d\mu \cdots \int_X f_\ell d\mu,$$

where the convergence is in $L^2(\mu)$. For $\ell = 1$, we say that $(a_1(n))_n$ is *ergodic*.

- *totally jointly ergodic for (X, \mathcal{B}, μ, T)* , if for all functions $f_1, \dots, f_\ell \in L^\infty(\mu)$, $W \in \mathbb{N}$, and $r \in \mathbb{Z}$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^{a_1(Wn+r)} f_1 \cdots T^{a_\ell(Wn+r)} f_\ell = \int_X f_1 d\mu \cdots \int_X f_\ell d\mu,$$

where the convergence is in $L^2(\mu)$. For $\ell = 1$, we say that $(a_1(n))_n$ is *totally ergodic*.¹

Studying (1) for $a_i(n) = in$, $1 \leq i \leq \ell$, Furstenberg managed to reprove (in [14]) Szemerédi’s theorem on arithmetic progressions.² In the same article, he also showed that the sequences $(n)_n, \dots, (\ell n)_n$ are (totally) jointly ergodic for every weakly mixing system.³

¹The notions of ‘‘ergodicity’’ and ‘‘total ergodicity’’ of a sequence are in general different. Let (X, \mathcal{B}, μ, T) be a system. T is *ergodic* (in which case we also call the system (X, \mathcal{B}, μ, T) *ergodic*) if whenever $T^{-1}A = A$, we have $\mu(A) \in \{0, 1\}$ (i.e., there are no T -invariant sets of non-trivial measure), and T is *totally ergodic* (where we also call the system (X, \mathcal{B}, μ, T) *totally ergodic*) if T^n is ergodic for all $n \in \mathbb{N}$. Taking an ergodic transformation T which is not totally ergodic, say T^k is not ergodic for some $k \in \mathbb{N}$, we have from von Neumann’s mean ergodic theorem that $(n)_n$ is an ergodic sequence, but $(kn + r)_n$ is not for any $r \in \mathbb{Z}$.

²Every dense subset of natural numbers contains arbitrarily long arithmetic progressions.

³We call a system (X, \mathcal{B}, μ, T) *weakly mixing* if T is *weakly mixing*, meaning that $T \times T$ is ergodic.

A few years later, Bergelson showed (in [1]) that for all essentially distinct integer polynomials p_1, \dots, p_ℓ ,⁴ the sequences $(p_1(n))_n, \dots, (p_\ell(n))_n$ are (totally) jointly ergodic for every weakly mixing system. With this result not only he extended Furstenberg's convergence result on weakly mixing systems, but also introduced the PET (i.e., Polynomial Exhaustion Technique) induction scheme, which led to a number of far-reaching extensions of Szemerédi's theorem with applications in various areas of mathematics.

It is natural to ask whether one can still get (total) joint ergodicity results by postulating less assumptions on the system while simultaneously strengthening the assumptions on the polynomial iterates. In this article, our main focus is to study total joint ergodicity properties for totally ergodic systems. To this end, it is convenient for us to have the following definition of independence of polynomials.

Definition. Let $V \subseteq \mathbb{R}$ with $0 \in V$ and $p_1, \dots, p_\ell \in \mathbb{R}[x]$. We say that p_1, \dots, p_ℓ are V -independent if $c_1 p_1 + \dots + c_\ell p_\ell \in \mathbb{Q}[x] + \mathbb{R}$, $c_i \in V$, $1 \leq i \leq \ell$, implies that $c_1 = \dots = c_\ell = 0$; otherwise, we call the p_i 's V -dependent.

Note that for any finite family of real polynomials p_1, \dots, p_ℓ , we have that

- (2) if p_1, \dots, p_ℓ are V -independent,
then $p_1(W \cdot + r), \dots, p_\ell(W \cdot + r)$ are V -independent for all $W \in \mathbb{N}$ and $r \in \mathbb{Z}$.

Frantzikinakis and Kra showed (in [11]) that if p_1, \dots, p_ℓ are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent integer polynomials,⁵ then $(p_1(n))_n, \dots, (p_\ell(n))_n$ are (totally) jointly ergodic for every totally ergodic system. Moreover, in that result, the independence assumption on the polynomial family is also necessary (see comments after [11, Theorem 1.1], and Theorem D below).⁶

Karageorgos and the first author showed (in [18]) that if p_1, \dots, p_ℓ are \mathbb{R} -independent real polynomials,⁷ then $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are (totally) jointly ergodic for every ergodic system.⁸

A conjecture⁹ of the first author of this article, following the philosophy of the results that were stated above, i.e., postulating a stronger assumption on the system while simultaneously assuming a weaker independence condition on the polynomial iterates, is that one should expect the following.

Conjecture 1. For $\ell \in \mathbb{N}$, let $p_1, \dots, p_\ell \in \mathbb{R}[x]$. p_1, \dots, p_ℓ are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent if, and only if, $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are totally jointly ergodic for every totally ergodic system.

Our first result is a positive answer to the sufficiency part of this conjecture.

Theorem A. For $\ell \in \mathbb{N}$, let $p_1, \dots, p_\ell \in \mathbb{R}[x]$. If p_1, \dots, p_ℓ are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent, then $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are totally jointly ergodic for every totally ergodic system.

⁴A polynomial $p \in \mathbb{Q}[x]$ is called *integer* if $p(\mathbb{Z}) \subseteq \mathbb{Z}$ (e.g., $p(n) = n(n+1)/2$). The non-constant polynomials p_1, \dots, p_ℓ are *essentially distinct* if for every $i \neq j$, $p_i - p_j$ is non-constant.

⁵In [11] such polynomials are called *rationally independent* (the independence condition there is presented in a different, equivalent, formulation).

⁶For characterizations of (total) joint ergodicity for integer polynomials, see also [4, 5, 12, 13]. Actually, all the results we stated up to this point were dealing with the notion of "joint ergodicity"; it is because of (2) that we can upgrade their conclusions to "total joint ergodicity".

⁷In [18] such polynomials are called *strongly independent*.

⁸See [19] for a generalization of this result for sequences of variable real polynomials.

⁹This conjecture was never recorded anywhere formally.

In fact, we prove a result more general than Theorem A, namely, Theorem E, which deals with combinations of polynomials and tempered Hardy field functions. Our proof relies on an improvement of a recent result of Frantzikinakis ([10, Theorem 1.9 (i)]—see Theorem 1.1) on joint ergodicity, which also implies a generalization of [2, Conjecture 6.1]. We defer the details to Subsection 1.2.

As we mentioned above, the necessity part of Conjecture 1 holds when p_1, \dots, p_ℓ are integer polynomials. In the case $\ell = 1$, we show that Conjecture 1 holds in general (see Corollary 3.7 which is derived from Theorem D). On the other hand, we (somewhat surprisingly) discover that the necessity part of Conjecture 1 is false in general. In fact, we obtain necessary and sufficient conditions for total joint ergodicity of two real polynomials p_1, p_2 with $p_1(0) = p_2(0) = 0$, which is the main result of the article.

Theorem B. *For all $p_1, p_2 \in \mathbb{R}[x]$ with $p_1(0) = p_2(0) = 0$, $([p_1(n)])_n, ([p_2(n)])_n$ are totally jointly ergodic for all totally ergodic systems if, and only if, the following holds:*

- (i) p_1, p_2 are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent; or
- (ii) $p_1 = f + cg$, $p_2 = u(f + (c + 1)g)$ for some $f, g \in \mathbb{Q}[x]$, where f is not a multiple of g ,¹⁰ $f(0) = g(0) = 0$, $g \not\equiv 0$, $c \in \mathbb{R} \setminus \mathbb{Q}$, with $u = \pm 1$, such that g is an integer polynomial.

We remark that $p_1(0) = p_2(0) = 0$ is an arguably common (and sometimes necessary) assumption in the study of multiple recurrence properties. However, it is natural to ask whether one can get a characterization for polynomials with non-trivial constant terms (see Question 2 below). Unlike the case of integer polynomials, the constant terms bring many additional difficulties in the study of the equidistribution properties for iterates of integer parts of real polynomials which are not integer ones. Indeed, it can happen that for two polynomials p_1, p_2 , in some totally ergodic system and for some $a \in \mathbb{R}$, $([p_1(n)])_n, ([p_2(n)])_n$ behave differently than $([p_1(n)])_n, ([p_2(n) + a])_n$ (see Proposition 7.1).

Theorem B indicates that a characteristic condition for more than two polynomials could be very intricate (see Problem 1). We also note that our method doesn't generalize to more than two terms (see Proposition 6.1, the argument of which works only for $\ell = 2$). Nevertheless, we were able to obtain a partial characterization in the general case.

Theorem C. *For $\ell \in \mathbb{N}$, let $p_1, \dots, p_\ell \in \mathbb{R}[x]$. If there exists a nonempty subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$ such that p_{i_1}, \dots, p_{i_k} are $\mathbb{R} \setminus \mathbb{Q}_*$ -dependent and all the irrational polynomials¹¹ in p_{i_1}, \dots, p_{i_k} , if any, are \mathbb{Q} -independent, then there exists a totally ergodic system (X, \mathcal{B}, μ, T) such that $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are not totally jointly ergodic for (X, \mathcal{B}, μ, T) .*

By combining Theorems A and C, we deduce that for two special classes of polynomials, the condition in Conjecture 1 is characteristic.

Theorem D. *For $\ell \in \mathbb{N}$, let $p_1, \dots, p_\ell \in \mathbb{Q}[x] + \mathbb{R}$ (resp. $p_1, \dots, p_\ell \in \mathbb{R}[x]$) so that all the irrational polynomials in p_1, \dots, p_ℓ , if any, are \mathbb{Q} -independent). p_1, \dots, p_ℓ are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent if, and only if, $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are totally jointly ergodic for every totally ergodic system.*

¹⁰By this we mean that there is no $s \in \mathbb{Q}$ such that $f = sg$.

¹¹By this we mean polynomials which don't belong to $\mathbb{Q}[x] + \mathbb{R}$.

Noticing that the independence condition for integer polynomials of Frantzikinakis and Kra is equivalent to $\mathbb{R} \setminus \mathbb{Q}_*$ -independence, we have that Theorem D generalizes the main theorem of [11] as well.

1.2. Improvement of a result on joint ergodicity. One of the main ingredients of this article is a recent criterion of Frantzikinakis ([10, Theorem 1.1]–see Theorem 2.4). In [10], Frantzikinakis provided an approach to study joint ergodicity properties without using either the Host-Kra theory of characteristic factors or equidistribution results on nilmanifolds, the combination of which is arguably the most influential approach to characterize the convergence of (1) to the expected limit when the sequences a_1, \dots, a_ℓ are coming from suitable classes of functions. Using this result, not only he obtained new convergence results, but also provided simpler proofs for most of the known ones.

As in [10], we assume that all Hardy fields \mathcal{H} considered (see Subsection 2.1 for the definition of a Hardy field and of functions in such a field) have the property:

$$(3) \quad \text{If } a, b \in \mathcal{H}, \text{ then } a \circ b^{-1} \in \mathcal{H}, \text{ and } a(\cdot + h) \in \mathcal{H} \text{ for all } h \geq 0.$$

For two functions $a, b : (x_0, \infty) \rightarrow \mathbb{R}$, we write $a \prec b$ if $|a(x)|/|b(x)| \rightarrow 0$ as $x \rightarrow \infty$.

Definition. We say that a Hardy field function $a : (x_0, \infty) \rightarrow \mathbb{R}$, $x_0 \geq 0$,

- (i) is *tempered* (and we write $a \in \mathcal{T}$) if there exists $k \in \mathbb{N} : x^{k-1} \log x \prec a(x) \prec x^k$.
- (ii) *stays logarithmically away from rational polynomials* if $\log x \prec a(x) - p(x)$ for all $p \in \mathbb{Q}[x]$.¹²

When the functions a_1, \dots, a_ℓ and their differences are in $\mathcal{T} + \mathcal{P}$,¹³ Frantzikinakis also showed the following result which resolves [2, Conjecture 6.1].

Theorem 1.1 (Theorem 1.9 (i), [10]). *For $\ell \in \mathbb{N}$, let $a_1, \dots, a_\ell : (x_0, +\infty) \rightarrow \mathbb{R}$ be functions from a Hardy field \mathcal{H} that satisfies (3). Suppose that the a_i 's and their differences are in $\mathcal{T} + \mathcal{P}$ and every non-trivial linear combination of them with at least one irrational coefficient, stays logarithmically away from rational polynomials. Then, the sequences $([a_1(n)])_n, \dots, ([a_\ell(n)])_n$ are jointly ergodic for every totally ergodic system.*

Following this newest approach of Frantzikinakis, and applying an extra twist in [10, Lemma 6.2] (see Proposition 3.1), we obtain the following improvement of the previous result.

Theorem E. *For $\ell \in \mathbb{N}$, let $a_1, \dots, a_\ell : (x_0, +\infty) \rightarrow \mathbb{R}$ be functions from a Hardy field \mathcal{H} that satisfies (3). Suppose that the a_i 's and their differences are in $\mathcal{T} + \mathcal{P}$ and every non-trivial linear combination of them with irrational or zero coefficients, stays logarithmically away from rational polynomials. Then, the sequences $([a_1(n)])_n, \dots, ([a_\ell(n)])_n$ are jointly ergodic for every totally ergodic system.*

¹²Or, equivalently, for all $p \in \mathbb{Q}[x] + \mathbb{R}$. Notice here the similarities between this condition and Weyl's criterion on equidistribution (in \mathbb{T}) when a is restricted to \mathcal{P} (see Remark 2.2 (ii)).

¹³As \mathcal{T} denotes the class of tempered functions, and \mathcal{P} the class of real polynomials, $\mathcal{T} + \mathcal{P}$ denotes the class of all linear combinations of tempered functions and real polynomials. The reason why it is natural to work with these classes, is that differences of them (i.e., derivatives) fall into the same classes, allowing one to study the corresponding averages by using variations of the PET induction, via van der Corput's lemma (approach that was initiated in [1]). For (total) joint ergodicity results for iterates coming from a Hardy field, see [2, 6, 7, 8, 22].

We prove Theorem E in Section 3. In particular, Theorem E implies Theorem A.

Example. To see why Theorem E improves Theorem 1.1, consider $p_1(n) = n^3 + an^2 + a^2n$, and $p_2(n) = n^2 + an$, for $a \in \mathbb{R} \setminus \mathbb{Q}$. Then, $p_1(n) - ap_2(n) = n^3 \in \mathbb{Q}[n]$, so Theorem 1.1 does not provide any info on the joint ergodicity of the sequences $([p_1(n)])_n, ([p_2(n)])_n$. On the other hand, if $b_1p_1(n) + b_2p_2(n) \in \mathbb{Q}[n]$, for $b_1, b_2 \in \mathbb{R} \setminus \mathbb{Q}_*$, we get $b_1 = b_2 = 0$, hence Theorem E implies that $([p_1(n)])_n, ([p_2(n)])_n$ are indeed jointly ergodic for every totally ergodic system.

Remark 1.2. Notice that if a function $a(x)$ stays logarithmically away from rational polynomials, then the same is true for the function $a(Wx + r)$ for all $W \in \mathbb{N}$, $r \in \mathbb{Z}$. So, in Theorems 1.1 and E (thus in the previous example as well) we actually have total joint ergodicity for the sequences of interest.

1.3. Questions and Problems. For Hardy field iterates coming from $\mathcal{T} + \mathcal{P}$, one can immediately ask the following question.

Question 1. What is a characteristic property in Theorem E so that the sequences $([a_1(n)])_n, \dots, ([a_\ell(n)])_n$ are (totally) jointly ergodic for every totally ergodic system?

As we mentioned before, we expect the answer to this question to be very hard as it is not clear how to answer it even for $\ell = 3$, when we deal with total joint ergodicity and we restrict our study to \mathcal{P} (see Problem 1 below).

Restricting to \mathcal{P} , while our method uses the assumption $p_1(0) = p_2(0) = 0$ crucially, it is reasonable to ask the following.

Question 2. Can the assumption $p_1(0) = p_2(0) = 0$ be dropped in Theorem B?

Since our approach to Theorem B cannot be extended to $\ell \geq 3$, one can state the following.

Problem 1. Extend Theorem B for $\ell \geq 3$.

While in general the notions of “ergodicity” and “total ergodicity” are different, we didn’t manage to find a pair of polynomial sequences that are jointly ergodic but not totally jointly ergodic.

Question 3. Does Theorem B characterize the notion of “joint ergodicity for all totally ergodic systems” for $([p_1(n)])_n, ([p_2(n)])_n$, where $p_1, p_2 \in \mathbb{R}[x]$ with $p_1(0) = p_2(0) = 0$?

In case the answer to the previous question is negative, a natural, follow-up problem is the following (which can be stated for any $\ell \geq 3$ sequences as well).

Problem 2. Similarly to Theorem B, find a characterization to the “joint ergodicity for all totally ergodic systems” notion.

1.4. Strategy and organization of the paper. We provide some background material in Section 2. In Section 3, we prove all the results stated in Section 1 except Theorem B. In fact, in Sections 2 and 3 we obtain stronger versions of these results for a fixed system (instead of all systems) and also deal with $W!$ -joint ergodic property (defined in Section 3 as well), and then derive the aforementioned results as corollaries. By applying an additional twist to the argument in [10, Lemma 6.2], we first prove Theorem E, an enhancement of

Frantzikinakis' result. This enables us to reduce Theorems A, C and D to the case where the system is a rotation on an abelian group; we then use Weyl's equidistribution theorem to prove them.

Sections 4, 5 and 6 are devoted to the proof of Theorem B. In Section 4, we prove some formulas for the integral of exponential functions over subtori of \mathbb{T}^2 and \mathbb{T}^3 . In Section 5, we first derive some equidistribution properties for polynomials, and then use the formulas obtained in Section 4 to study limits of exponential sums, as

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^2 t_i [p_i(n)]\right).$$

Using the tools developed in the previous sections, we prove Theorem B by splitting it into various cases in Section 6.

Finally, in Section 7 (the appendix), we provide two explicit examples to demonstrate the complexity of the behavior of joint ergodicity for $([p_1(n)])_n, ([p_2(n)])_n$ with $p_1, p_2 \in \mathbb{R}[x]$.

Notation. With $\mathbb{N} = \{1, 2, \dots\}$, \mathbb{Z} , \mathbb{Q} , \mathbb{R} , and \mathbb{C} we denote the sets of natural, integer, rational, real, and complex numbers respectively. For $m \in \mathbb{N}$, $\mathbb{T}^m = (\mathbb{R}/\mathbb{Z})^m$ denotes the m dimensional torus and $(a(n))_n$ denotes a sequence indexed over the natural numbers. For every $A \subseteq \mathbb{R}$, we write $A_* := A \setminus \{0\}$. $[\cdot]$ (resp. $\{\cdot\}$) is the integer part (resp. fractional part) function, where, for all $x \in \mathbb{R}$, we have $x = [x] + \{x\}$. Finally, $e(t) := e^{2\pi it}$, $t \in \mathbb{R}$.

2. AN IMPROVEMENT OF FRANTZIKINAKIS' RESULT

2.1. Hardy field functions. Let R be the collection of equivalence classes of real valued functions defined on some halfline (x_0, ∞) , $x_0 \geq 0$, where two functions that agree eventually are identified. These classes are called *germs* of functions. A *Hardy field* is a subfield of the ring $(R, +, \cdot)$ that is closed under differentiation. Here, we use the word *function* when we refer to elements of R (understanding that all the operations defined and statements made for elements of R are considered only for sufficiently large $x \in \mathbb{R}$).

As it was mentioned in the introduction, following [10], we assume that all Hardy fields \mathcal{H} considered satisfy the property:

$$\text{If } a, b \in \mathcal{H}, \text{ then } a \circ b^{-1} \in \mathcal{H}, \text{ and } a(\cdot + h) \in \mathcal{H} \text{ for all } h \geq 0.^{14}$$

In particular, we will deal with Hardy field functions form $\mathcal{T} + \mathcal{P}$, where we recall that \mathcal{T} denotes the class of tempered functions (i.e., functions a that, for some $k \in \mathbb{N}$, satisfy $x^{k-1} \log x \prec a(x) \prec x^k$), and \mathcal{P} is the class of real polynomials.

2.2. Equidistribution on (sub)tori. Let Y be a finite-dimensional subtorus (i.e., a subgroup of a finite-dimensional torus). We say that a sequence $(x_n)_n \subseteq Y$ is *equidistributed* on Y if for every complex-valued Riemann integrable function f on Y we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N f(x_n) = \int_Y f(x) dm_Y(x),$$

¹⁴Such a Hardy field is the one of Pfaffian functions which contains all the logarithmico-exponential functions (a is a *logarithmico-exponential Hardy field function* if it belongs to a Hardy field of real valued functions and it is defined on some $(x_0, +\infty)$, $x_0 \geq 0$, by a finite combination of symbols $+$, $-$, \times , \div , $\sqrt[\cdot]{\cdot}$, \exp , \log acting on the real variable x and on real constants). For more on Hardy field functions, see [3, 7, 8, 16].

where m_Y is the Haar measure on Y .

The following is a consequence of a special case of [15, Theorem 1.9] (see also [21]).

Proposition 2.1. *Let $m, r \in \mathbb{N}$, $v_i = (v_{i,1}, \dots, v_{i,m}) \in \mathbb{Q}^m$ for $1 \leq i \leq r$, H be the set of $(u_1, \dots, u_m) \in \mathbb{R}^m$ such that $u_1 v_{i,1} + \dots + u_m v_{i,m} = 0$ for all $1 \leq i \leq r$, Y be the set of $(\{x_1\}, \dots, \{x_m\}) \in \mathbb{T}^m$ for some $(x_1, \dots, x_m) \in H$, and $p = (p_1, \dots, p_m): \mathbb{Z} \rightarrow \mathbb{R}^m$ be a vector valued polynomial taking values in H . If $(\{p(n)\})_n$ is not equidistributed on Y , then there exist $k = (k_1, \dots, k_m) \in \mathbb{Z}^m$ which does not belong to $\text{span}_{\mathbb{Q}}\{v_1, \dots, v_r\}$ ¹⁵ and some $t \in \mathbb{R}$ such that $k_1 p_1(n) + \dots + k_m p_m(n) \equiv t \pmod{1}$ for all $n \in \mathbb{Z}$.*

Remark 2.2. (i) *If we impose the additional assumption $p(0) = (0, \dots, 0)$ in Proposition 2.1, then we have that $k_1 p_1(n) + \dots + k_m p_m(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.*

(ii) *Proposition 2.1 also extends to the case where $r = 0$, $H = \mathbb{R}^m$ and $Y = \mathbb{T}^m$ ($\text{span}_{\mathbb{Q}}\{v_1, \dots, v_r\}$ is understood as the singleton $\{(0, \dots, 0)\}$). This is in fact Weyl's criterion (see [23], or [20]): For any sequence $(x_n)_n \subseteq \mathbb{R}^m$, $(\{x_n\})_n$ ¹⁶ is equidistributed on \mathbb{T}^m if, and only if, for all $k \in \mathbb{Z}^m \setminus \{(0, \dots, 0)\}$, we have that $\frac{1}{N} \sum_{n=1}^N e^{2\pi i \langle k, x_n \rangle} \rightarrow 0$, as $N \rightarrow \infty$, where $\langle \cdot, \cdot \rangle$ denotes the standard inner product.*

2.3. Frantzikinakis' criterion on joint ergodicity. As we mentioned in the introduction, the proof of Theorem E (as the one of Theorem 1.1) is primarily based on Frantzikinakis' main result from [10] ([10, Theorem 1.1]—see Theorem 2.4 below). In order to state it we need some additional definitions.

Following [17], given a system (X, \mathcal{B}, μ, T) , we inductively define the *Gowers-Host-Kra seminorms* $\| \cdot \|_k$:¹⁷ For $f \in L^\infty(\mu)$, we let

$$\|f\|_0 := \int f \, d\mu, \text{ and for } k \in \mathbb{N}, \|f\|_k^{2^k} := \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \|\bar{f} \cdot T^n f\|_{k-1}^{2^{k-1}}.$$

Next, we define the *spectrum* of a transformation.

Definition. For a system (X, \mathcal{B}, μ, T) we let

$$\text{Spec}(T) := \{t \in [0, 1) : Tf = e(t)f \text{ for some nonzero } f \in L^2(\mu)\}.$$

Remark 2.3. *Notice that, for a totally ergodic transformation T , $\text{Spec}(T) \subseteq [0, 1) \setminus \mathbb{Q}_*$, as T cannot have an eigenvalue $\lambda \neq 1$ which is a root of unity.*

The following notions of “good for seminorm estimates” and “good for equidistribution” characterize joint ergodicity for ergodic systems.

Definition. For a collection of sequences $a_1, \dots, a_\ell : \mathbb{N} \rightarrow \mathbb{Z}$, and a system (X, \mathcal{B}, μ, T) , we say that $(a_1(n), \dots, a_\ell(n))_n$ is

- (i) *good for seminorm estimates* for (X, \mathcal{B}, μ, T) , if there exists $s \in \mathbb{N}$ such that if $f_1, \dots, f_\ell \in L^\infty(\mu)$ and $\|f_{i_0}\|_s = 0$ for some $1 \leq i_0 \leq \ell$, then

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^{a_1(n)} f_1 \dots T^{a_{i_0}(n)} f_{i_0} = 0,$$

¹⁵ $\text{span}_A\{v_1, \dots, v_r\}$ denotes all the linear combinations of v_1, \dots, v_r with coefficients from A , $A \subseteq \mathbb{R}$.

¹⁶If $x_n = (x_1(n), \dots, x_m(n))$, by $(\{x_n\})_n$ we mean $(\{x_1(n)\}, \dots, \{x_m(n)\})_n$.

¹⁷ $\| \cdot \|_0$, which is not a seminorm, is defined for convenience.

where the convergence is in $L^2(\mu)$. It is *good for seminorm estimates*, if it is good for seminorm estimates for every ergodic system.

- (ii) *good for equidistribution* for (X, \mathcal{B}, μ, T) , if for all $t_1, \dots, t_\ell \in \text{Spec}(T)$, not all of them 0, we have

$$(4) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e(t_1 a_1(n) + \dots + t_\ell a_\ell(n)) = 0.$$

It is *good for equidistribution*, if it is good for equidistribution for every system, or, equivalently, if (4) holds for all $t_1, \dots, t_\ell \in [0, 1)$, not all of them 0.

Theorem 2.4 (Theorem 1.1, [10]). *For $\ell \in \mathbb{N}$, let $a_1, \dots, a_\ell : \mathbb{N} \rightarrow \mathbb{Z}$ be sequences. $(a_1(n))_n, \dots, (a_\ell(n))_n$ are jointly ergodic for an ergodic system (X, \mathcal{B}, μ, T) if, and only if, $(a_1(n), \dots, a_\ell(n))_n$ is good for seminorm estimates and equidistribution for (X, \mathcal{B}, μ, T) .*

For the class of Hardy field functions from $\mathcal{T} + \mathcal{P}$, we have the following result which follows from [10, Proposition 6.5].

Proposition 2.5 (Proposition 6.5, [10]). *For $\ell \in \mathbb{N}$, let $a_1, \dots, a_\ell : (x_0, \infty) \rightarrow \mathbb{R}$ be functions from a Hardy field that satisfies (3) such that the a_i 's and their pairwise differences are non-constant functions in $\mathcal{T} + \mathcal{P}$. Then $([a_1(n)], \dots, [a_\ell(n)])_n$ is good for seminorm estimates.*

Remark 2.6. *To show our joint ergodicity results, by Theorem 2.4, via Proposition 2.5, it only suffices to verify the “good for equidistribution” property, which is a Weyl-type sum, hence, we only have to check equidistribution on (sub)tori.*

In particular, for Hardy field functions of at most polynomial growth,¹⁸ hence for functions from \mathcal{T} as well, we have the following result of Boshernitzan.

Theorem 2.7 ([3]). *Let $a : (x_0, \infty) \rightarrow \mathbb{R}$ be a Hardy field function with at most polynomial growth. $(a(n))_n$ is equidistributed on \mathbb{T} if, and only if, it stays logarithmically away from rational polynomials.*

2.4. The improved condition on joint ergodicity. Using Theorem 2.7, Theorem E will follow from Proposition 2.8 (see below). The latter provides a sufficient condition of (total) joint ergodicity for a specific system (its proof follows the philosophy of [10, Lemma 6.2]). While our assumption is weaker than that of [10, Lemma 6.2] (we restrict the coefficients of the linear combinations to the spectrum of the transformation T and not the whole set of irrational numbers), our conclusion is stronger.

It is more convenient for us, instead of dealing with $\text{Spec}(T)$, to work with the set

$$S(T) := \text{Spec}(T) + \mathbb{Z}.$$

Notice that, because of Remark 2.3, $S(T) \subseteq (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Z}$.

Proposition 2.8. *Let (X, \mathcal{B}, μ, T) be a totally ergodic system and $a_1, \dots, a_\ell : (x_0, +\infty) \rightarrow \mathbb{R}$ be functions from a Hardy field that satisfies (3). Suppose that the a_i 's and their differences are in $\mathcal{T} + \mathcal{P}$ and that every non-trivial linear combination of them, with coefficients from $S(T) \setminus \mathbb{Z}_*$, stays logarithmically away from rational polynomials. Then, the sequences $([a_1(n)])_n, \dots, ([a_\ell(n)])_n$ are totally jointly ergodic for (X, \mathcal{B}, μ, T) .*

¹⁸A function $a : (x_0, \infty) \rightarrow \mathbb{R}$ has *at most polynomial growth* if there exists $k \in \mathbb{N}$ such that $a(x) \prec x^k$.

To show Proposition 2.8, we follow the arguments of [10, Lemma 6.2]. In our proof, we delete the t_i 's that are equal to 0 at the beginning, and then continue working under the assumption that all the t_i 's are nonzero (in which case the claim holds and leads to the required strengthening of [10, Proposition 6.4] and Theorem 1.1).

Proof of Proposition 2.8. We first claim that for any $s \leq \ell$, $t_1, \dots, t_s \in S(T) \setminus \mathbb{Z}$, and for any Riemann integrable function $G: \mathbb{T}^s \rightarrow \mathbb{C}$, we have that

$$(5) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^s t_i a_i(n)\right) G(a_1(n), \dots, a_s(n)) = 0.$$

To see this, approximating G from below and above by continuous functions, and then by trigonometric polynomials, we may assume without loss of generality that $G(x_1, \dots, x_s) = e(k_1 x_1 + \dots + k_s x_s)$, $x_1, \dots, x_s \in \mathbb{T}$ for some $k_1, \dots, k_s \in \mathbb{Z}$.

The left-hand side of (5) equals to

$$(6) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^s (t_i + k_i) a_i(n)\right).$$

Clearly, each $t_i + k_i$ belongs to $S(T) \setminus \mathbb{Z}$. Since every non-trivial linear combination of a_1, \dots, a_ℓ , with coefficients from $S(T) \setminus \mathbb{Z}_*$, stays logarithmically away from rational polynomials, we have from Theorem 2.7 that (6) equals to 0 by Weyl's equidistribution criterion.

To prove the statement, by Remark 2.6, it suffices to show that for all $t_1, \dots, t_\ell \in S(T) \setminus \mathbb{Z}_*$ not all equal to 0, we have that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^{\ell} t_i [a_i(n)]\right) = 0.$$

We may assume without loss of generality that $t_1, \dots, t_m \neq 0$ and $t_{m+1} = \dots = t_\ell = 0$ for some $1 \leq m \leq \ell$. Then, applying the claim for $s = m$ and $G(x_1, \dots, x_m) := e(-\{x_1\}t_1 - \dots - \{x_m\}t_m)$, $x_1, \dots, x_m \in \mathbb{T}$, we have that

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^{\ell} t_i [a_i(n)]\right) &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^m t_i [a_i(n)]\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^m t_i a_i(n)\right) e\left(-\sum_{i=1}^m t_i \{a_i(n)\}\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^m t_i a_i(n)\right) G(a_1(n), \dots, a_m(n)) = 0, \end{aligned}$$

as was to be shown. \square

3. TOTAL JOINT ERGODICITY FOR SPECIAL CLASSES OF POLYNOMIALS

In this section, we prove Theorems E, A, C, and D. We first provide in Subsection 3.1 variations of the last three theorems for a fixed system (as we did in Proposition 2.8 for Theorem E), and then derive the desired results in Subsection 3.2.

3.1. Total joint ergodicity for a fixed totally ergodic system. We start with an implication of Proposition 2.8. Restricting to \mathcal{P} , we get the following.

Proposition 3.1. *Let $\ell \in \mathbb{N}$, (X, \mathcal{B}, μ, T) be a totally ergodic system, and $p_1, \dots, p_\ell \in \mathbb{R}[x]$. If p_1, \dots, p_ℓ are $S(T) \setminus \mathbb{Z}_*$ -independent, then $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are totally jointly ergodic for (X, \mathcal{B}, μ, T) .*

For $W \in \mathbb{N}$, we now define the notion of W -joint ergodicity.

Definition. For $\ell \in \mathbb{N}$, a collection of sequences $a_1, \dots, a_\ell : \mathbb{N} \rightarrow \mathbb{Z}$, and $W \in \mathbb{N}$, we say that $(a_1(n))_n, \dots, (a_\ell(n))_n$ are W -jointly ergodic for the system (X, \mathcal{B}, μ, T) , if for all $f_1, \dots, f_\ell \in L^\infty(\mu)$, we have

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N T^{a_1(Wn)} f_1 \cdots T^{a_\ell(Wn)} f_\ell = \int_X f_1 d\mu \cdots \int_X f_\ell d\mu,$$

where the convergence takes place in $L^2(\mu)$. For $\ell = 1$, we say that $(a_1(n))_n$ is W -ergodic.

Proposition 3.2. *Let $\ell \in \mathbb{N}$, (X, \mathcal{B}, μ, T) be a totally ergodic system, and $p_1, \dots, p_\ell \in \mathbb{R}[x]$. If there exists a nonempty subset $\{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$ such that p_{i_1}, \dots, p_{i_k} are $S(T) \setminus \mathbb{Z}_*$ -dependent and all the irrational polynomials in p_{i_1}, \dots, p_{i_k} , if any, are \mathbb{Q} -independent, then there exists $W_0 \equiv W_0(p_1, \dots, p_\ell) \in \mathbb{N}$ such that for all $W \geq W_0$, $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are not $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) .*

Proof. By setting the functions f_i to be constant 1 for polynomials outside of the set $\{p_{i_1}, \dots, p_{i_k}\}$, we may assume without loss of generality that $\{i_1, \dots, i_k\} = \{1, \dots, \ell\}$. Then, there exist $c_i \in S(T) \setminus \mathbb{Z}_*$, not all of them 0 with

$$(7) \quad c_1 p_1 + \cdots + c_\ell p_\ell = q \in \mathbb{Q}[x] + \mathbb{R}.$$

We may assume without loss of generality that $p_1, \dots, p_m \notin \mathbb{Q}[x] + \mathbb{R}$ and $p_{m+1}, \dots, p_\ell \in \mathbb{Q}[x] + \mathbb{R}$. By the assumption, we have that p_1, \dots, p_m are \mathbb{Q} -independent. Multiplying both sides of (7) by an integer if necessary, we may assume without loss of generality that $q \in \mathbb{Z}[x] + \mathbb{R}$. It is clear that there exists $W_0 \equiv W_0(p_1, \dots, p_\ell) \in \mathbb{N}$ such that for all $W \geq W_0$, we have that $p_i(W!n) \in \mathbb{Z}[n] + \mathbb{R}$, $m+1 \leq i \leq \ell$. It suffices to show that

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^{\ell} c_i [p_i(W!n)]\right) \neq 0.$$

Note that the left-hand side of the previous relation is equal to

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(q(W!n) - \sum_{i=1}^{\ell} c_i \{p_i(W!n)\}\right) = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(-\sum_{i=1}^m c_i \{p_i(W!n)\}\right) \cdot v,$$

where $v = e\left(q(0) - \sum_{i=m+1}^{\ell} c_i \{p_i(0)\}\right)$.

Let $F: \mathbb{T}^m \rightarrow \mathbb{C}$, with $F(x_1, \dots, x_m) = e\left(-\sum_{i=1}^m c_i x_i\right)$. Then F is a Riemann integrable function. Since p_1, \dots, p_m are \mathbb{Q} -independent, the same is true for $p_1(W!), \dots, p_m(W!)$.

By Weyl's criterion, $(\{p_1(W!n)\}, \dots, \{p_m(W!n)\})_n$ is equidistributed on \mathbb{T}^m . So,

$$\begin{aligned} \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(-\sum_{i=1}^m c_i \{p_i(W!n)\}\right) &= \int_{[0,1]^m} F(x_1, \dots, x_m) d(x_1, \dots, x_m) \\ &= \int_{[0,1]^m} e\left(-\sum_{i=1}^m c_i x_i\right) d(x_1, \dots, x_m) \\ &= \prod_{i=1}^m \int_{[0,1]} e(-c_i x) dx. \end{aligned}$$

For all $1 \leq i \leq m$, since $c_i \notin \mathbb{Z}_*$, we have that $\int_{[0,1]} e(-c_i x) dx = 1$ if $c_i = 0$ and $\int_{[0,1]} e(-c_i x) dx = \frac{1-e(-c_i)}{2\pi i c_i} \neq 0$ if $c_i \notin \mathbb{Z}$; the proof is complete. \square

Combining the previous two results, we get the following result.

Corollary 3.3. *Let $\ell \in \mathbb{N}$, (X, \mathcal{B}, μ, T) be a totally ergodic system, and $p_1, \dots, p_\ell \in \mathbb{Q}[x] + \mathbb{R}$ (resp. $p_1, \dots, p_\ell \in \mathbb{R}[x]$) so that all the irrational polynomials in p_1, \dots, p_ℓ , if any, are \mathbb{Q} -independent). Then the following are equivalent:*

- (i) $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are totally jointly ergodic for (X, \mathcal{B}, μ, T) .
- (ii) There exists $W_0 \equiv W_0(p_1, \dots, p_\ell) \in \mathbb{N}$ such that $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) for all $W \geq W_0$.
- (iii) There exists an infinite set $I \equiv I(p_1, \dots, p_\ell) \subseteq \mathbb{N}$ such that $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) for all $W \in I$.
- (iv) p_1, \dots, p_ℓ are $S(T)$ -independent (resp. $S(T) \setminus \mathbb{Z}_*$ -independent).

Proof. The implications (i) \Rightarrow (ii) \Rightarrow (iii) are immediate, while the implications (iii) \Rightarrow (iv) and (iv) \Rightarrow (i) follow from Proposition 3.2 and Proposition 3.1 respectively. (When $p_1, \dots, p_\ell \in \mathbb{Q}[x] + \mathbb{R}$, $S(T) \setminus \mathbb{Z}_*$ -independence is equivalent to $S(T)$ -independence.) \square

Remark 3.4. *By setting $\ell = 1$ to the previous result, we get a characterization of when $([p(n)])_n$, where $p \in \mathbb{R}[x]$, is totally ergodic for a specific totally ergodic system.*

Notice that Corollary 3.3 also implies the result of Frantzikinakis and Kra ([11, Theorem 1.1]) which was presented in the introduction (in which joint ergodicity and total joint ergodicity are equivalent). Indeed, if $\{p_1, \dots, p_\ell\}$ are rationally independent integer polynomials (in which case of course they belong to $\mathbb{Q}[x] + \mathbb{R}$), then they are $S(T)$ -independent (since $S(T)$ consists of irrational numbers and 0). So, by the previous corollary, we have total joint ergodicity, hence joint ergodicity.

3.2. Total joint ergodicity for all totally ergodic systems. We are now ready to use results from Subsections 2.4 and 3.1 to prove Theorems A, C, D and E; we start with Theorem E.

Proof of Theorem E. Let (X, \mathcal{B}, μ, T) be a totally ergodic system. Since every non-trivial linear combination of a_1, \dots, a_ℓ , with coefficients from $\mathbb{R} \setminus \mathbb{Q}_*$, stays logarithmically away from rational polynomials, the same is true for coefficients from $S(T) \setminus \mathbb{Z}_*$. By Proposition 2.8, $([a_1(n)])_n, \dots, ([a_\ell(n)])_n$ are totally jointly ergodic for (X, \mathcal{B}, μ, T) . \square

Immediate implication of the previous result is Theorem A.

Proof of Theorem A. The result follows from Theorem E since when we restrict to \mathcal{P} , “stays logarithmically away from rational polynomials” is equivalent to not be in $\mathbb{Q}[x] + \mathbb{R}$. \square

To prove Theorem C, we need the following lemma.

Lemma 3.5. *For any $\ell \in \mathbb{N}$ and $c_1, \dots, c_\ell \in \mathbb{R}$, there exist $D \in \mathbb{N}$ and a totally ergodic system (X, \mathcal{B}, μ, T) such that $Dc_1, \dots, Dc_\ell \in S(T)$. In particular, if c_1, \dots, c_ℓ and 1 are \mathbb{Q} -independent, we can choose T so that $S(T) = \text{span}_{\mathbb{Z}}\{1, c_1, \dots, c_\ell\}$.*

Proof. If all the c_i 's are rational numbers, then we may take X to be the trivial system and D to be the product of the denominators of the c_i 's. So, we assume that at least one of the c_i 's is irrational. It is not hard to see that there exist $a_1, \dots, a_m \in \mathbb{R}$ for some $1 \leq m \leq \ell$ such that $a_1, \dots, a_m, 1$ are \mathbb{Q} -independent, and that each of the c_1, \dots, c_ℓ is a linear combination of $a_1, \dots, a_m, 1$ with rational coefficients. So, there exists $D \in \mathbb{N}$ such that each of Dc_1, \dots, Dc_ℓ is a linear combination of $a_1, \dots, a_m, 1$ with integer coefficients. Hence, it suffices to construct a totally ergodic system with $\{a_1\}, \dots, \{a_m\} \in \text{Spec}(T)$.

Let $X = \mathbb{T}^m$ be endowed with the Haar measure and let $T: X \rightarrow X$, with

$$T(x_1, \dots, x_m) = (x_1 + a_1, \dots, x_m + a_m).$$

We have that $\{a_1\}, \dots, \{a_m\} \in \text{Spec}(T)$. Moreover, since $a_1, \dots, a_m, 1$ are \mathbb{Q} -independent, it is not hard to see that this system is totally ergodic.

The “in particular” part is straightforward from the proof (in which case $\{c_1, \dots, c_\ell\} = \{a_1, \dots, a_m\}$). \square

Proof of Theorem C. Let $\{i_1, \dots, i_k\} \subseteq \{1, \dots, \ell\}$ be such that p_{i_1}, \dots, p_{i_k} are not $\mathbb{R} \setminus \mathbb{Q}_*$ -independent and all the irrational polynomials in p_{i_1}, \dots, p_{i_k} (if any) are \mathbb{Q} -independent. Assume that $c_1 p_{i_1} + \dots + c_k p_{i_k} \in \mathbb{Q}[x] + \mathbb{R}$ for some $c_1, \dots, c_k \in \mathbb{R} \setminus \mathbb{Q}_*$. By Lemma 3.5, there exists a totally ergodic system (X, \mathcal{B}, μ, T) and $D \in \mathbb{N}$ such that $Dc_1, \dots, Dc_k \in S(T)$. Then $Dc_1 p_{i_1} + \dots + Dc_k p_{i_k} \in \mathbb{Q}[x] + \mathbb{R}$. Since $c_1, \dots, c_k \notin \mathbb{Q}_*$, we have that $Dc_1, \dots, Dc_k \in S(T) \setminus \mathbb{Z}_*$. By Proposition 3.2, there exists $W_0 \in \mathbb{N}$ depending only on p_1, \dots, p_ℓ such that $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are not $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) for all $W \geq W_0$, from where the result follows. \square

The following is a corollary of Theorems A and C.

Corollary 3.6. *For $\ell \in \mathbb{N}$, let $p_1, \dots, p_\ell \in \mathbb{Q}[x] + \mathbb{R}$ (resp. $p_1, \dots, p_\ell \in \mathbb{R}[x]$) so that all the irrational polynomials in p_1, \dots, p_ℓ , if any, are \mathbb{Q} -independent). Then the following are equivalent:*

- (i) $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are totally jointly ergodic for every totally ergodic system.
- (ii) There exists $W_0 \equiv W_0(p_1, \dots, p_\ell) \in \mathbb{N}$ such that for any $W \geq W_0$, $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are $W!$ -jointly ergodic for every totally ergodic system.
- (iii) There exists an infinite set $I \equiv I(p_1, \dots, p_\ell) \subseteq \mathbb{N}$ so that for any $W \in I$, $([p_1(n)])_n, \dots, ([p_\ell(n)])_n$ are $W!$ -jointly ergodic for every totally ergodic system.
- (iv) p_1, \dots, p_ℓ are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent.

Proof. It is clear that (i) \Rightarrow (ii) \Rightarrow (iii). The implication (iii) \Rightarrow (iv) follows from the proof of Theorem C. Finally, the implication (iv) \Rightarrow (i) follows from Theorem A. \square

Proof of Theorem D. Follows immediately by the previous corollary. \square

By letting $\ell = 1$ in Theorem D, we get that Conjecture 1 holds for a single real polynomial.

Corollary 3.7. *Let $p \in \mathbb{R}[x]$. p is $\mathbb{R} \setminus \mathbb{Q}_*$ -independent¹⁹ if, and only if, $([p(n)])_n$ is totally ergodic for every totally ergodic system.*

While for $p_1, \dots, p_\ell \in \mathbb{Q}[x] + \mathbb{R}$ (resp. $p_1, \dots, p_\ell \in \mathbb{R}[x]$ so that all the irrational polynomials in p_1, \dots, p_ℓ , if any, are \mathbb{Q} -independent) we have that $W!$ -joint ergodicity for infinitely many W 's is equivalent to total joint ergodicity (result that holds for a single arbitrary $p \in \mathbb{R}[x]$ as well), for two general polynomial iterates $p_1, p_2 \in \mathbb{R}[x]$ we can have that $W!$ -joint ergodicity for a co-finite set doesn't even imply joint ergodicity (see Proposition 7.2).

4. INTEGRALS OF EXPONENTIAL FUNCTIONS ON SUBTORI

In this section we will prove two statements, Propositions 4.2 and 4.3, that will help us deal with equidistribution results on subtori. In particular, the integrals that we are computing in these two statements have connection with exponential sums, and will be used in the proof of our main result, Theorem B, later in the article.

We first introduce some helpful notation: For $(x, y) \in \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}_*\}$ we let

$$\left(\frac{x}{y}\right)_* := \begin{cases} x/y, & y \neq 0 \\ 1, & x = y = 0 \end{cases}.$$

In particular, for $(x, y) \in \mathbb{R}^2 \setminus \{(x, 0) : x \in \mathbb{R}_*\}$,

$$\left(\frac{x}{y}\right)_* \neq 0 \Leftrightarrow x \neq 0 \text{ or } y = 0.$$

Recall that $e(x) = e^{2\pi i x}$, $x \in \mathbb{R}$. Using the previous notation, skipping the trivial computations, we have the following lemma.

Lemma 4.1. *For all $\alpha \in \mathbb{R}$, $N \in \mathbb{N}$, and $t \in \mathbb{R}_*$, we have that*

$$(8) \quad \frac{1}{N} \sum_{n=0}^{N-1} e(\alpha n) = \left(\frac{e(\alpha N) - 1}{N(e(\alpha) - 1)}\right)_*,$$

and

$$(9) \quad \frac{1}{t} \int_0^t e(\alpha x) dx = \left(\frac{e(\alpha t) - 1}{2\pi i \alpha t}\right)_*.$$

In particular, (8) is nonzero if, and only if, $\alpha \notin (\mathbb{Z}/N) \setminus \mathbb{Z}$, and (9) is nonzero if, and only if, $\alpha \notin (\mathbb{Z}/t)_$, where, for $s \in \mathbb{R}_*$, we set $\mathbb{Z}/s := \{a \in \mathbb{R} : as \in \mathbb{Z}\}$.*

In the rest of the paper, we will use Lemma 4.1 freely without citations. We start with an estimate for the integral of exponential functions along one dimensional subtori of \mathbb{T}^2 .

¹⁹I.e., p cannot be written as $p = cq$, where $q \in \mathbb{Q}[x] + \mathbb{R}$ and $c \in \mathbb{R} \setminus \mathbb{Q}_*$.

Proposition 4.2. *Let $\alpha, \beta \in \mathbb{R}$, $a, b \in \mathbb{Z}_*$ with $\gcd(a, b) = 1$, $Y = \{(\{x\}, \{y\}) : (x, y) \in \mathbb{R}^2, ax + by = 0\}$, and m_Y be the Haar measure on Y . Then*

$$\int_Y e(\alpha x + \beta y) dm_Y(x, y) \neq 0$$

if, and only if, the following conditions hold: $\frac{\alpha}{a} \notin (\mathbb{Z}/a) \setminus \mathbb{Z}$, $\frac{\beta}{b} \notin (\mathbb{Z}/b) \setminus \mathbb{Z}$, $\frac{\alpha}{a} - \frac{\beta}{b} \notin \mathbb{Z}_$.*

Proof. Let a, b have the same sign, say $a, b > 0$.²⁰ Since $ax + by = 0$, we have $x = bt$, and $y = -at$, for some $t \in \mathbb{R}$. So

$$\begin{aligned} \int_Y e(\alpha x + \beta y) dm_Y(x, y) &= \int_0^1 e(\alpha\{bt\} + \beta\{-at\}) dt \\ &= \sum_{j=0}^{ab-1} \int_{\frac{j}{ab}}^{\frac{j+1}{ab}} e(\alpha\{bt\} + \beta\{-at\}) dt \\ &= \sum_{j=0}^{ab-1} \int_0^{\frac{1}{ab}} e\left(\alpha\left\{bt + \frac{j}{a}\right\} + \beta\left\{-at - \frac{j}{b}\right\}\right) dt \\ &= \sum_{j=0}^{ab-1} \int_0^{\frac{1}{ab}} e\left(\alpha\{bt\} + \beta\{-at\} + \alpha\left\{\frac{j}{a}\right\} + \beta\left\{-\frac{j}{b}\right\} - \beta\right) dt \\ &= \sum_{j=0}^{ab-1} \int_0^{\frac{1}{ab}} e\left(\alpha bt - \beta at + \alpha\left\{\frac{j}{a}\right\} + \beta\left\{-\frac{j}{b}\right\}\right) dt \\ &= \sum_{j=0}^{ab-1} e\left(\alpha\left\{\frac{j}{a}\right\} + \beta\left\{-\frac{j}{b}\right\}\right) \cdot \frac{1}{ab} \left(\frac{e\left(\frac{\alpha}{a} - \frac{\beta}{b}\right) - 1}{2\pi i \left(\frac{\alpha}{a} - \frac{\beta}{b}\right)}\right)_*. \end{aligned}$$

By the Chinese remainder theorem, $\left(\left\{\frac{j}{a}\right\}, \left\{-\frac{j}{b}\right\}\right)$, $0 \leq j \leq ab - 1$, takes all the values of $\left(\left\{\frac{m}{a}\right\}, \left\{-\frac{n}{b}\right\}\right)$, $0 \leq m \leq a - 1$, $0 \leq n \leq b - 1$, and so it takes each value exactly once. Then

$$\begin{aligned} \sum_{j=0}^{ab-1} e\left(\alpha\left\{\frac{j}{a}\right\} + \beta\left\{-\frac{j}{b}\right\}\right) &= \sum_{k=0}^{a-1} \sum_{j=0}^{b-1} e\left(\alpha\left\{\frac{k}{a}\right\}\right) \cdot e\left(\beta\left\{-\frac{j}{b}\right\}\right) \\ &= e(\beta) \sum_{k=0}^{a-1} \sum_{j=0}^{b-1} e\left(\alpha \cdot \frac{k}{a}\right) \cdot e\left(-\beta \cdot \frac{j}{b}\right) \\ &= e(\beta) ab \left(\frac{e(\alpha) - 1}{a\left(e\left(\frac{\alpha}{a}\right) - 1\right)}\right)_* \cdot \left(\frac{e(-\beta) - 1}{b\left(e\left(-\frac{\beta}{b}\right) - 1\right)}\right)_*. \end{aligned}$$

²⁰The case where a, b have different signs is very similar and gives the same result modulo a nonzero constant.

To sum up, we have that

$$\int_Y e(\alpha x + \beta y) dm_Y(x, y) \neq 0 \text{ if, and only if,}$$

$$\frac{\alpha}{a} \notin (\mathbb{Z}/a) \setminus \mathbb{Z}, \frac{\beta}{b} \notin (\mathbb{Z}/b) \setminus \mathbb{Z}, \frac{\alpha}{a} - \frac{\beta}{b} \notin \mathbb{Z}_*,$$

as was to be shown. \square

We next estimate the integral of exponential functions along one dimensional subtori of \mathbb{T}^3 . Such estimates turn out to be very difficult to compute. We only provide an estimate for a special case which will be used in our approach.

Proposition 4.3. *Let $\alpha, \beta \in \mathbb{R}$, $w \in \mathbb{Z}$, $a, b, r \in \mathbb{Z}_*$ with $\gcd(a, b) = 1$, $Y = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, ax + by = rx + bz = 0\}$, and m_Y be the Haar measure on Y . Then*

$$\int_Y e(\alpha x + \beta y + wz) dm_Y(x, y, z) \neq 0$$

if, and only if, the following conditions hold:

$$-\frac{\alpha}{a} + \frac{\beta}{b} + \frac{rw}{ab} \notin \mathbb{Z}_*, -\frac{\alpha}{a} + \frac{rwb^*}{a} \notin (\mathbb{Z}/a) \setminus \mathbb{Z}, \frac{\beta}{b} + \frac{rwa^*}{b} \notin (\mathbb{Z}/b) \setminus \mathbb{Z},$$

where a^* is the unique element in $[1, b-1]$ such that $aa^* \equiv 1 \pmod{b}$, and b^* is the unique element in $[1, a-1]$ such that $bb^* \equiv 1 \pmod{a}$.

Proof. Let $a, b, r > 0$.²¹ Since $ax + by = rx + bz = 0$, we may write $x = -bt$, $y = at$, and $z = rt$, for some $t \in \mathbb{R}$. So, using the fact that w is an integer, we get

$$\begin{aligned} & \int_Y e(\alpha x + \beta y + wz) dm_Y(x, y, z) \\ &= \int_0^1 e(\alpha\{-bt\} + \beta\{at\} + w\{rt\}) dt \\ &= \sum_{j=0}^{ab-1} \int_{\frac{j}{ab}}^{\frac{j+1}{ab}} e(\alpha\{-bt\} + \beta\{at\} + w\{rt\}) dt \\ &= \sum_{j=0}^{ab-1} \int_0^{\frac{1}{ab}} e\left(\alpha\left\{-bt - \frac{j}{a}\right\} + \beta\left\{at + \frac{j}{b}\right\} + w\left\{rt + \frac{rj}{ab}\right\}\right) dt \\ &= \sum_{j=0}^{ab-1} \int_0^{\frac{1}{ab}} e\left(\alpha\{-bt\} + \beta\{at\} + w\{rt\} + \alpha\left\{-\frac{j}{a}\right\} + \beta\left\{\frac{j}{b}\right\} + w\left\{\frac{rj}{ab}\right\} - \alpha\right) dt \\ &= \sum_{j=0}^{ab-1} \int_0^{\frac{1}{ab}} e\left(\left(-\alpha b + \beta a + rw\right)t + \alpha\left\{-\frac{j}{a}\right\} + \beta\left\{\frac{j}{b}\right\} + w\left\{\frac{rj}{ab}\right\}\right) dt \\ &= \left(\frac{e\left(-\frac{\alpha}{a} + \frac{\beta}{b} + \frac{rw}{ab}\right) - 1}{2\pi i \left(-\frac{\alpha}{a} + \frac{\beta}{b} + \frac{rw}{ab}\right)}\right)_* \cdot \frac{1}{ab} \sum_{j=0}^{ab-1} e\left(\alpha\left\{-\frac{j}{a}\right\} + \beta\left\{\frac{j}{b}\right\} + w\left\{\frac{rj}{ab}\right\}\right). \end{aligned}$$

²¹We can assume that $r > 0$ (this follows by the fact that $w \in \mathbb{Z}$); the case where a, b have different signs is very similar and gives the same result modulo a nonzero constant.

As we saw in Proposition 4.2, for all $0 \leq j \leq ab - 1$, there exist unique $0 \leq m \leq a - 1$, $0 \leq n \leq b - 1$, so that

$$\left(\left\{-\frac{j}{a}\right\}, \left\{\frac{j}{b}\right\}\right) = \left(\left\{-\frac{m}{a}\right\}, \left\{\frac{n}{b}\right\}\right).$$

Hence, j is the unique integer in $[0, ab - 1]$ such that $j \equiv m \pmod{a}$ and $j \equiv n \pmod{b}$, and so

$$j \equiv mbb^* + naa^* \pmod{ab},$$

where a^* is the unique element in $[1, b - 1]$ such that $aa^* \equiv 1 \pmod{b}$, and b^* is the unique element in $[1, a - 1]$ such that $bb^* \equiv 1 \pmod{a}$. We have that

$$\left\{\frac{rj}{ab}\right\} = \left\{\frac{r(mbb^* + naa^*)}{ab}\right\},$$

so

$$\begin{aligned} & \frac{1}{ab} \sum_{j=0}^{ab-1} e\left(\alpha\left\{-\frac{j}{a}\right\} + \beta\left\{\frac{j}{b}\right\} + w\left\{\frac{rj}{ab}\right\}\right) \\ &= \frac{1}{ab} \sum_{m=0}^{a-1} \sum_{n=0}^{b-1} e\left(\alpha\left\{-\frac{m}{a}\right\} + \beta\left\{\frac{n}{b}\right\} + w\left\{\frac{r(mbb^* + naa^*)}{ab}\right\}\right) \\ &= \frac{1}{ab} \sum_{m=0}^{a-1} \sum_{n=0}^{b-1} e\left(m\left(-\frac{\alpha}{a} + \frac{rwb^*}{a}\right) + n\left(\frac{\beta}{b} + \frac{rwa^*}{b}\right) - \alpha\right) \\ &= e(-\alpha) \left(\frac{e(-\alpha + rwb^*) - 1}{a(e(-\frac{\alpha}{a} + \frac{rwb^*}{a}) - 1)}\right) \cdot \left(\frac{e(\beta + rwa^*) - 1}{b(e(\frac{\beta}{b} + \frac{rwa^*}{b}) - 1)}\right). \end{aligned}$$

Putting everything together, we have that

$$\int_Y e(\alpha x + \beta y + wz) dm_Y(x, y, z) \neq 0 \text{ if, and only if,}$$

$$-\frac{\alpha}{a} + \frac{\beta}{b} + \frac{rw}{ab} \notin \mathbb{Z}_*, \quad -\frac{\alpha}{a} + \frac{rwb^*}{a} \notin (\mathbb{Z}/a) \setminus \mathbb{Z}, \quad \frac{\beta}{b} + \frac{rwa^*}{b} \notin (\mathbb{Z}/b) \setminus \mathbb{Z},$$

as was to be shown. \square

5. ESTIMATING AVERAGES OF EXPONENTIAL SUMS

5.1. Properties of polynomial orbits. For a fixed triple of real polynomials (p_1, p_2, p_3) , with $p_1(0) = p_2(0) = p_3(0) = 0$, we define a subtorus $Y = Y(p_1, p_2, p_3)$, so that the sequence $(\{p_1(W!n)\}, \{p_2(W!n)\}, \{p_3(W!n)\})_n$ is equidistributed on Y for large enough W (see the definition before Proposition 2.1).

For such a triple of polynomials, let

$$K(p_1, p_2, p_3) := \{(k_1, k_2, k_3) \in \mathbb{Z}^3 : k_1 p_1 + k_2 p_2 + k_3 p_3 \in \mathbb{Q}[x]\},$$

and

$$Y(p_1, p_2, p_3) := \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, k_1 x + k_2 y + k_3 z = 0, \\ (k_1, k_2, k_3) \in K(p_1, p_2, p_3)\}.$$

We remark that, for all $W \in \mathbb{N}$, we have $K(p_1(W\cdot), p_2(W\cdot), p_3(W\cdot)) = K(p_1, p_2, p_3)$ and $Y(p_1(W\cdot), p_2(W\cdot), p_3(W\cdot)) = Y(p_1, p_2, p_3)$.

We have the following equidistribution property.

Lemma 5.1. *Let $p_1, p_2, p_3 \in \mathbb{R}[x]$ be polynomials with $p_1(0) = p_2(0) = p_3(0) = 0$. If $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$, then it is also equidistributed on $Y(p_1, p_2, p_3)$.*

Proof. Suppose that this is not the case. Note that $Y(p_1, p_2, p_3)$ is a subtorus of \mathbb{T}^3 . By Proposition 2.1, there exists $(k_1, k_2, k_3) \in \mathbb{Z}^3$ which does not belong to the \mathbb{Q} -span of $K(p_1, p_2, p_3)$ such that $k_1 p_1(n) + k_2 p_2(n) + k_3 p_3(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$. However, this implies that $(k_1, k_2, k_3) \in K(p_1, p_2, p_3)$, a contradiction by definition. Hence $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ is equidistributed on $Y(p_1, p_2, p_3)$. \square

Remark 5.2. *A rather strong restriction in Lemma 5.1 is that it is only applicable when $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$. In general, $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values on finitely many shifted copies of $Y(p_1, p_2, p_3)$. For example, if $p_1(n) = -\alpha n$, $p_2(n) = (\alpha + 1/2)n$ and $p_3(n) = \beta n$ for some $\alpha, \beta \in \mathbb{R}$ with $\alpha, \beta, 1$ being \mathbb{Q} -independent, then*

$$Y(p_1, p_2, p_3) := \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, x + y = 0\}.$$

However, $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ does not lie in $Y(p_1, p_2, p_3)$, but in $Y(p_1, p_2, p_3) \cup (Y(p_1, p_2, p_3) + (0, 1/2, 0) \bmod \mathbb{Z}^3)$.

To ensure that we can use Lemma 5.1 for our purposes, we need another result (see Lemma 5.3 below). To state it, we need the following notation.

Let K_0 be a basis of $K(p_1, p_2, p_3)$. For each $(k_1, k_2, k_3) \in K_0$, suppose that

$$k_1 p_1 + k_2 p_2 + k_3 p_3 = g$$

for some polynomial g in $\mathbb{Q}[x]$. If $Q(k_1, k_2, k_3)$ denotes the smallest natural number such that $g(Q(k_1, k_2, k_3)!n)/\gcd(k_1, k_2, k_3)$ is integer-valued, we also let

$$(10) \quad W_0(p_1, p_2, p_3) := \min_{\substack{K_0 \text{ basis of} \\ K(p_1, p_2, p_3)}} \max_{(k_1, k_2, k_3) \in K_0} Q(k_1, k_2, k_3).$$

Lemma 5.3. *Let $p_1, p_2, p_3 \in \mathbb{R}[x]$ be polynomials with $p_1(0) = p_2(0) = p_3(0) = 0$. If $W \geq W_0(p_1, p_2, p_3)$, where W_0 is given in (10), then the sequence $(\{p_1(W!n)\}, \{p_2(W!n)\}, \{p_3(W!n)\})_n$ takes values in $Y(p_1, p_2, p_3)$.*

Proof. Let K_0 be a basis of $K(p_1, p_2, p_3)$ and fix $(k_1, k_2, k_3) \in K_0$. Then $k_1 p_1 + k_2 p_2 + k_3 p_3 = g \in \mathbb{Q}[x]$. For convenience denote $Q := Q(k_1, k_2, k_3)$. Let $a_1, a_2, a_3 \in \mathbb{Z}$ be such that $k_1 a_1 + k_2 a_2 + k_3 a_3 = \gcd(k_1, k_2, k_3)$. Then $\{p_i(Q!n)\} = \left\{ p_i(Q!n) - \frac{a_i g(Q!n)}{\gcd(k_1, k_2, k_3)} \right\}$ for $1 \leq i \leq 3$. Moreover,

$$\sum_{i=1}^3 k_i \left(p_i(Q!n) - \frac{a_i g(Q!n)}{\gcd(k_1, k_2, k_3)} \right) = g(Q!n) - g(Q!n) = 0.$$

So, $(\{p_1(Q!n)\}, \{p_2(Q!n)\}, \{p_3(Q!n)\})$ belongs to $Y(p_1, p_2, p_3)$. \square

As it was mentioned in previous sections, it is important for our study to consider the case of two rational polynomials where one is, or is not, a multiple of the other. In the latter case, we have the following helpful lemma.

Lemma 5.4. *Let $f, g \in \mathbb{Q}[x]$, $f(0) = g(0) = 0$, $g \not\equiv 0$, and suppose that f is not a multiple of g . If $af + bg \in \mathbb{Q}[x]$ for some $a, b \in \mathbb{R}$, then we must have that $a, b \in \mathbb{Q}$.*

Proof. Suppose that $f(n) = \sum_{i=1}^K f_i n^i$ and $g(n) = \sum_{i=1}^K g_i n^i$, for some $K \in \mathbb{N}$ and $f_i, g_i \in \mathbb{Q}$. Then $af_i + bg_i \in \mathbb{Q}$ for all $1 \leq i \leq K$. If $b \notin \mathbb{Q}$, then, since $g_{i'} \neq 0$ for some $1 \leq i' \leq K$ and $f_{i'}, g_{i'} \in \mathbb{Q}$, we must have that $a \notin \mathbb{Q}$. Since $a, b \notin \mathbb{Q}$ and $af_i + bg_i \in \mathbb{Q}$ for all $1 \leq i \leq K$, we must also have that $f_i g_j - f_j g_i = 0$ for all $1 \leq i, j \leq K$, which implies that f is a multiple of g , a contradiction. Hence, $b \in \mathbb{Q}$, which in turn implies that $af \in \mathbb{Q}[x]$, so $a \in \mathbb{Q}$ too (since f , being not a multiple of g , is not constant zero). \square

5.2. Characterizing when an exponential sum is zero. The purpose of this subsection is to study the average

$$(11) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^2 t_i [p_i(n)]\right),$$

where $t_1, t_2 \in \mathbb{R}$. For convenience denote by $p_3 := t_1 p_1 + t_2 p_2$. For two special types of polynomial sequences (p_1, p_2, p_3) , we will compute the subtorus $Y(p_1, p_2, p_3)$, and partially characterize when (11) is equal to 0.

The first type of (p_1, p_2, p_3) we are interested in is given in the following proposition.

Proposition 5.5. *Let $f, g \in \mathbb{Q}[x]$, $f(0) = g(0) = 0$, $g \not\equiv 0$, f is not a multiple of g , $c \in \mathbb{R} \setminus \mathbb{Q}$, $u_1, u_2 \in \mathbb{Z}_*$ with $\gcd(u_1, u_2) = 1$, and $d \in \mathbb{Q}_*$. Let*

$$p_1 = u_1(f + cg), \quad p_2 = u_2(f + (c + d)g),$$

$t_1, t_2 \in (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Z}$, not both in \mathbb{Z} , and $p_3 = t_1 p_1 + t_2 p_2$. We have:

(i) *If $t_1 u_1 + t_2 u_2 \notin \mathbb{Q}$, then*

$$Y(p_1, p_2, p_3) = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_2 x - u_1 y = 0\},$$

and (11) is zero if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$;

(ii) *If $t_1 u_1 + t_2 u_2 \in \mathbb{Z}$, and $t_2, c, 1$ are \mathbb{Q} -independent, then*

$$Y(p_1, p_2, p_3) = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_2 x - u_1 y = 0\},$$

and (11) is zero if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$;

(iii) *If $t_1 u_1 + t_2 u_2 \in \mathbb{Z}$, and $t_2 = ac + b$ for some $a, b \in \mathbb{Q}$, then*

$$Y(p_1, p_2, p_3) = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_2 x - u_1 y = au_2 dx - u_1 z = 0\}.$$

Moreover, if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$ and $au_2 d \in \mathbb{Z}$, then (11) is nonzero if, and only if, $\frac{ad}{u_1} \notin \mathbb{Z}_$.*

Remark 5.6. *As the corresponding t_i 's will be taken in $S(T)$, for a totally ergodic transformation, (i), (ii) and (iii) cover all possible cases since $S(T) \subseteq (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Z}$.*

Notice that in case (iii), even if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$, we were only able to provide an estimate for (11) when $au_2 d \in \mathbb{Z}$. However, this is good enough for our purposes.

Proof of Proposition 5.5. For convenience, we denote by

$$K := K(p_1, p_2, p_3), \quad \text{and} \quad Y := Y(p_1, p_2, p_3).$$

Note that (11) becomes

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^2 t_i [p_i(n)]\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\left\{\sum_{i=1}^2 t_i p_i(n)\right\} - \sum_{i=1}^2 t_i \{p_i(n)\}\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(-t_1 \{p_1(n)\} - t_2 \{p_2(n)\} + \{p_3(n)\}\right). \end{aligned}$$

If $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in Y , then by Lemma 5.1, we have that

$$(12) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^2 t_i [p_i(n)]\right) = \int_Y e(-t_1 x - t_2 y + z) d\mu_Y(x, y, z).$$

In order to analyze the description of Y , we need to compute K . Recall that for $(k_1, k_2, k_3) \in \mathbb{Q}^3$, we have that $(k_1, k_2, k_3) \in K$ if

$$(13) \quad \begin{aligned} k_1 p_1 + k_2 p_2 + k_3 p_3 &= ((k_1 u_1 + k_2 u_2 + k_3(t_1 u_1 + t_2 u_2))c + k_3 t_2 u_2 d + k_2 u_2 d)g \\ &\quad + (k_1 u_1 + k_2 u_2 + k_3(t_1 u_1 + t_2 u_2))f \in \mathbb{Q}[n]. \end{aligned}$$

Since f is not a multiple of g , then by Lemma 5.4, (13) holds if, and only if,

$$(14) \quad k_3(t_1 u_1 + t_2 u_2), (k_1 u_1 + k_2 u_2 + k_3(t_1 u_1 + t_2 u_2))c + k_3 t_2 u_2 d \in \mathbb{Q}.$$

We first prove part (i). If $t_1 u_1 + t_2 u_2 \notin \mathbb{Q}$, then (14) holds if, and only if, $k_3 = k_1 u_1 + k_2 u_2 = 0$. In this case, K is generated by the vector $(u_2, -u_1, 0)$. Therefore,

$$(15) \quad Y = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_2 x - u_1 y = 0\}.$$

Then (12) is 0 since $\int_0^1 e(z) dz = 0$. This concludes the first part.

For part (ii), if $t_1 u_1 + t_2 u_2 \in \mathbb{Z}$, then we may replace t_i by $t_i + m_i$ for some integer m_i (notice that this does not change the sum or the conditions in parts (ii) or (iii)). Since $\gcd(u_1, u_2) = 1$, we may assume that $t_1 u_1 + t_2 u_2 = 0$. (Indeed, if $t_1 u_1 + t_2 u_2 = k$, since $\gcd(u_1, u_2) = 1$, by the Euclidean algorithm, there are integers m_1, m_2 such that $m_1 u_1 + m_2 u_2 = 1$, so, $km_1 u_1 + km_2 u_2 = k$. Hence $(t_1 - km_1)u_1 + (t_2 - km_2)u_2 = 0$.) So, (14) becomes

$$(16) \quad (k_1 u_1 + k_2 u_2)c + k_3 t_2 u_2 d \in \mathbb{Q}.$$

Suppose that $t_2, c, 1$ are \mathbb{Q} -independent. Since $u_2 d \neq 0$, (16) holds if, and only if, $k_3 = k_1 u_1 + k_2 u_2 = 0$. Again in this case, K is generated by the vector $(u_2, -u_1, 0)$. As before, Y is given by (15), and so (12) is equal to 0. This concludes part (ii).

For part (iii), suppose that $t_2 = ac + b$, for some $a, b \in \mathbb{Q}$ with $au_2 d \in \mathbb{Z}$. If $t_2 \in \mathbb{Z}$, then, since $t_1 u_1 + t_2 u_2 = 0$ and $t_1 \in (\mathbb{R} \setminus \mathbb{Q}) \cup \mathbb{Z}$, we have to have $t_1 \in \mathbb{Z}$, a contradiction. Hence, we have that $t_2 \notin \mathbb{Q}$ (which also implies that $t_1 \notin \mathbb{Q}$). Then, (16) is equivalent to

$$k_1 u_1 + k_2 u_2 + k_3 a u_2 d = 0.$$

Therefore, K is generated by the vectors $(u_2, -u_1, 0)$ and $(au_2d, 0, -u_1)$. So,

$$Y = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_2x - u_1y = au_2dx - u_1z = 0\}.$$

Since $au_2d \in \mathbb{Z}_*$, setting $a = u_2, b = -u_1, r = au_2d, \alpha = -t_1, \beta = -t_2$ and $w = 1$ in Proposition 4.3, we have that (12) is nonzero if, and only if, the following hold:

- $\frac{ad}{u_1} \notin \mathbb{Z}_*$;
- $\frac{t_1}{u_2} + (-u_1)^* ad \notin (\mathbb{Z}/u_2) \setminus \mathbb{Z}$;
- $\frac{t_2}{u_1} - \frac{u_2^* au_2d}{u_1} \notin (\mathbb{Z}/u_1) \setminus \mathbb{Z}$.

Since $t_1, t_2 \notin \mathbb{Q}$, the last two conditions are always true. So, the average is nonzero if, and only if, $\frac{ad}{u_1} \notin \mathbb{Z}_*$. This proves part (iii) and completes the proof. \square

The second type of (p_1, p_2, p_3) we are interested in is given in the following proposition.

Proposition 5.7. *Let $g \in \mathbb{Q}[x]$, $g(0) = 0$, $g \neq 0$, $c \in \mathbb{R} \setminus \mathbb{Q}$, $u_1, u_2 \in \mathbb{Z}_*$ with $\gcd(u_1, u_2) = 1$, and $d \in \mathbb{Q}_*$. Let*

$$p_1 = u_1cg, \quad p_2 = u_2(c+d)g,$$

$t_1, t_2 \in \mathbb{R}$, not both in \mathbb{Z} , and $p_3 = t_1p_1 + t_2p_2$. We have:

(i) *If $t_1u_1c + t_2u_2(c+d)$, c and 1 are \mathbb{Q} -independent, then*

$$Y(p_1, p_2, p_3) = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_2x - u_1y = 0\}.$$

Moreover, if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$, then (11) is zero;

(ii) *If $t_1u_1c + t_2u_2(c+d) = sc + t$ for some $s, t \in \mathbb{Q}$, then*

$$Y(p_1, p_2, p_3) = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_1y - u_2x = u_1z - sx = 0\}.$$

Moreover, if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in $Y(p_1, p_2, p_3)$ and $s \in \mathbb{Z}$, then (11) is nonzero if, and only if, the following hold:

- $\frac{t-t_2u_2d}{u_1u_2c} \notin \mathbb{Z}_*$;
- $\frac{t_1}{u_2} + \frac{s(-u_1)^*}{u_2} \notin (\mathbb{Z}/u_2) \setminus \mathbb{Z}$;
- $\frac{t_2}{u_1} - \frac{su_2^*}{u_1} \notin (\mathbb{Z}/u_1) \setminus \mathbb{Z}$.

Remark 5.8. *As with Proposition 5.5 (iii), in case (ii) of the previous statement, we were only able to provide an estimate for (11) for the special case where $s \in \mathbb{Z}$, which suffices for our purposes.*

Proof of Proposition 5.7. For convenience, we denote by

$$K := K(p_1, p_2, p_3), \quad \text{and} \quad Y := Y(p_1, p_2, p_3).$$

Similar to the proof of Proposition 5.5, if $(\{p_1(n)\}, \{p_2(n)\}, \{p_3(n)\})_n$ takes values in Y , then

$$(17) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\sum_{i=1}^2 t_i [p_i(n)]\right) = \int_Y e(-t_1x - t_2y + z) d\mu_Y(x, y, z).$$

In order to analyze the description of Y , we need to compute K . Note that for $(k_1, k_2, k_3) \in \mathbb{Q}^3$, we have $(k_1, k_2, k_3) \in K$ if, and only if,

$$(18) \quad (k_1u_1 + k_2u_2)c + k_3(t_1u_1c + t_2u_2(c+d)) \in \mathbb{Q}.$$

We first prove part (i). Since $t_1u_1c + t_2u_2(c + d)$, c and 1 are \mathbb{Q} -independent, we have that (18) holds if, and only if,

$$k_3 = k_1u_1 + k_2u_2 = 0.$$

Therefore, K is generated by the vector $(u_2, -u_1, 0)$. So, Y is the same as case (i) of Proposition 5.5, and (11) equals 0. This concludes part (i).

We next prove part (ii). Since c is irrational, (18) holds if, and only if,

$$k_1u_1 + k_2u_2 + k_3s = 0.$$

Therefore, K is generated by the vectors $(u_2, -u_1, 0)$ and $(s, 0, -u_1)$. So, we have that

$$Y(p_1, p_2, p_3) = \{(\{x\}, \{y\}, \{z\}) : (x, y, z) \in \mathbb{R}^3, u_1y - u_2x = u_1z - sx = 0\}.$$

By Proposition 4.3 for $a = -u_2, b = u_1, r = -s, \alpha = -t_1, \beta = -t_2$, and $w = 1$, (17) is nonzero if, and only if, the following hold:

- $\frac{t_1}{u_2} + \frac{t_2}{u_1} - \frac{s}{u_1u_2} = \frac{t-t_2u_2d}{u_1u_2c} \notin \mathbb{Z}_*$;
- $\frac{t_1}{u_2} + \frac{s(-u_1)^*}{u_2} \notin (\mathbb{Z}/u_2) \setminus \mathbb{Z}$;
- $\frac{t_2}{u_1} - \frac{su_2^*}{u_1} \notin (\mathbb{Z}/u_1) \setminus \mathbb{Z}$.

This completes the proof of the statement. \square

6. CHARACTERIZING TOTAL JOINT ERGODICITY FOR TWO TERMS

In this section, we prove our main result on polynomial iterates, i.e., Theorem B.

6.1. Type-B pairs. Our first step is to reduce the total joint ergodicity problem to the case of some special pairs which are closely related to the pairs appearing in Theorem B case (ii), which we refer as ‘‘Type-B’’.

Definition. Let (X, \mathcal{B}, μ, T) be a system. We say that the pair (p_1, p_2) of real polynomials is of *Type-B for the system* (X, \mathcal{B}, μ, T) if there exist $f, g \in \mathbb{Q}[x]$, $f(0) = g(0) = 0$, $g \not\equiv 0$, $c \in S(T) \setminus \mathbb{Z}$, $u_1, u_2 \in \mathbb{Z}_*$ with $\gcd(u_1, u_2) = 1$, and $d \in \mathbb{Q}_*$, such that

$$(19) \quad p_1 = u_1(f + cg), \quad p_2 = u_2(f + (c + d)g).$$

Proposition 6.1. *Let (X, \mathcal{B}, μ, T) be a totally ergodic system. If p_1, p_2 are $S(T) \setminus \mathbb{Z}_*$ -dependent with $p_1(0) = p_2(0) = 0$, then either*

- (i) *there exists $W_0 \equiv W_0(p_1, p_2) \in \mathbb{N}$ such that $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) for all $W \geq W_0$; or*
- (ii) *(p_1, p_2) is of Type-B for (X, \mathcal{B}, μ, T) .*

Proof. If $p_1 \in \mathbb{Q}[x]$ and $p_2 \notin \mathbb{Q}[x]$, in which case p_2 is \mathbb{Q} -independent (the case $p_2 \in \mathbb{Q}[x]$ and $p_1 \notin \mathbb{Q}[x]$ is analogous), or $p_1, p_2 \in \mathbb{Q}[x]$, or $p_1, p_2 \notin \mathbb{Q}[x]$ and p_1, p_2 are \mathbb{Q} -independent, then, by Proposition 3.2, there exists $W_0 \in \mathbb{N}$ depending only on p_1, p_2 such that $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) for all $W \geq W_0$.

The remaining case is when $p_1, p_2 \notin \mathbb{Q}[x]$ and p_1, p_2 are \mathbb{Q} -dependent. Then, we have

$$(20) \quad c_1p_1 + c_2p_2 = q, \quad d_1p_1 + d_2p_2 = q'$$

for some $q, q' \in \mathbb{Q}[x]$, $c_1, c_2 \in S(T) \setminus \mathbb{Z}_*$ not both 0, and $d_1, d_2 \in \mathbb{Z}_*$ with $\gcd(d_1, d_2) = 1$ (since $p_1, p_2 \notin \mathbb{Q}[x]$). We remark that d_1, d_2 and q' are independent of $S(T)$. For convenience we let $r := \det \begin{bmatrix} c_1 & c_2 \\ d_1 & d_2 \end{bmatrix}$.

There exists $W_0 \in \mathbb{N}$ depending on p_1, p_2 , such that for all $W \geq W_0$ and $n \in \mathbb{N}$, $q(W!n)$ is an integer, and $q'(W!n)$ is an integer which is divisible by $d_1 d_2$.²²

To study the $W!$ -joint ergodicity property, we have to consider the following average:

$$(21) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e \left(\sum_{i=1}^2 c_i [p_i(W!n)] \right),$$

which equals to

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e \left(\sum_{i=1}^2 c_i p_i(W!n) - \sum_{i=1}^2 c_i \{p_i(W!n)\} \right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e \left(-c_1 \{p_1(W!n)\} - c_2 \{p_2(W!n)\} \right). \end{aligned}$$

Since $p_1(W!\cdot) \notin \mathbb{Q}[x]$, and $d_1 p_1(W!n) + d_2 p_2(W!n) = q'(W!n) \in d_1 d_2 \mathbb{Z}$, we have that the last line of the previous relation is equal to

$$(22) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e \left(-c_1 \{p_1(W!n)\} - c_2 \left\{ -\frac{d_1}{d_2} p_1(W!n) \right\} \right).$$

For every n , we have that $\left(\{p_1(W!n)\}, \left\{ -\frac{d_1}{d_2} p_1(W!n) \right\} \right)$ takes value in the subtorus

$$Y := \{(\{x\}, \{y\}) : (x, y) \in \mathbb{R}^2, d_1 x + d_2 y = 0\}.$$

Note that $p_1(0) = p_2(0) = 0$. If $\left(\{p_1(W!n)\}, \left\{ -\frac{d_1}{d_2} p_1(W!n) \right\} \right)_n$ is not equidistributed on Y then, by Proposition 2.1, there exist $(k_1, k_2) \in \mathbb{Z}^2 \setminus \text{span}_{\mathbb{Q}}\{(d_1, d_2)\}$ such that

$$k_1 p_1(W!n) + k_2 \left(-\frac{d_1}{d_2} p_1(W!n) \right) \in \mathbb{Z}.$$

Since $p_1(W!n) \notin \mathbb{Z}$ for some n , we have that $k_1 d_2 = k_2 d_1$, a contradiction to the assumption that $(k_1, k_2) \notin \text{span}_{\mathbb{Q}}\{(d_1, d_2)\}$. So $\left(\{p_1(W!n)\}, \left\{ -\frac{d_1}{d_2} p_1(W!n) \right\} \right)_n$ is equidistributed on Y .

Let $F: \mathbb{T}^2 \rightarrow \mathbb{C}$ be the function $F(x, y) = e(-c_1 x - c_2 y)$. Then F is Riemann integrable and (22) is equal to

$$(23) \quad \int_Y e(-c_1 x - c_2 y) dm_Y(x, y).$$

Since at least one of the c_1, c_2 is irrational, we have that either both are irrational, or one of them is irrational and the other is 0. In both cases, By Proposition 4.2, (23), and

²²This is another reason why we assume that p_1, p_2 take the value 0 at 0. If some of q, q' was constant, then the last claim would not be true in general.

hence (21) too, is nonzero if $r \notin \mathbb{Q}_*$, in which case $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) .

The remaining case is when (20) holds and $r \in \mathbb{Q}_*$. Note that $c_1, c_2 \notin \mathbb{Q}$.

Let $g := \frac{q'}{d_1 r}$ and $f := -\frac{q}{r}$. By (20),

$$p_1 = \frac{\det \begin{bmatrix} q & c_2 \\ q' & d_2 \end{bmatrix}}{r} = \frac{\det \begin{bmatrix} q & (c_1 d_2 - r)/d_1 \\ q' & d_2 \end{bmatrix}}{r} = -d_2 \left(f + \left(c_1 - \frac{r}{d_2} \right) g \right),$$

and

$$p_2 = \frac{\det \begin{bmatrix} c_1 & q \\ d_1 & q' \end{bmatrix}}{r} = d_1 (f + c_1 g).$$

To complete the proof that (p_1, p_2) is of Type-B for (X, \mathcal{B}, μ, T) , it suffices to check that q' , and thus g , is not constant (hence 0, since $p_1(0) = p_2(0) = 0$). Suppose on the contrary that $q' \equiv 0$. Then, by (20), we have $p_1 = -\frac{d_2}{d_1} p_2$. Again by (20), $q = c_1 p_1 + c_2 p_2 = -\frac{r}{d_1} p_2$, which is impossible since $q \in \mathbb{Q}[x]$ and $p_2 \notin \mathbb{Q}[x]$. \square

6.2. Proof of Theorem B. In this subsection, we prove Theorem B. Rescaling relevant coefficients and polynomials if necessary, it is not hard to see that condition (ii) in Theorem B is equivalent to the following:

- (ii)' $p_1 = u_1(f + cg), p_2 = u_2(f + (c + d)g)$ for some $f, g \in \mathbb{Q}[x]$, where f is not a multiple of g , $f(0) = g(0) = 0$, $g \neq 0$, $c \in \mathbb{R} \setminus \mathbb{Q}$, $u_1, u_2 \in \mathbb{Z}$ with $u_1, u_2 = \pm 1$, and $d \in \mathbb{Q}_*$ such that dg is an integer polynomial.

In the rest of this section, we will work with condition (ii)' instead of (ii) as it is notation-wise closer to the one of Type-B pairs.

Since Theorem B deals with total joint ergodicity, analogously to (11) and (21) that we introduced in Subsections 5.2 and 6.1 respectively, for $W \in \mathbb{N}$ and $r \in \mathbb{Z}$, we will deal with

$$(24) \quad \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e \left(\sum_{i=1}^2 t_i [p_i(Wn + r)] \right).$$

6.2.1. Sufficiency. We first prove the sufficiency part of Theorem B. Assume that both of its conditions fail. We start by reducing the problem to the Type-B case. By the failure of condition (i), p_1, p_2 are $\mathbb{R} \setminus \mathbb{Q}_*$ -dependent, so $c_1 p_1 + c_2 p_2 \in \mathbb{Q}[x]$ for some $c_1, c_2 \in \mathbb{R} \setminus \mathbb{Q}_*$, not both equal to 0.

By Lemma 3.5, there exists $D \in \mathbb{N}$ and a totally ergodic system (X, \mathcal{B}, μ, T) such that $Dc_1, Dc_2 \in S(T)$. Since $Dc_1, Dc_2 \notin \mathbb{Q}_*$, we have that p_1, p_2 are $S(T) \setminus \mathbb{Q}_*$ -dependent. By Proposition 6.1, either $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$ -jointly ergodic for (X, \mathcal{B}, μ, T) for some sufficiently large W depending on p_1, p_2 , and thus $([p_1(n)])_n, ([p_2(n)])_n$ are not totally jointly ergodic for (X, \mathcal{B}, μ, T) , or there exist $f, g \in \mathbb{Q}[x]$, $f(0) = g(0) = 0$, $g \neq 0$, $c \in S(T) \setminus \mathbb{Z}$, $u_1, u_2 \in \mathbb{Z}_*$ with $\gcd(u_1, u_2) = 1$, and $d \in \mathbb{Q}_*$ such that

$$p_1 = u_1(f + cg), \quad p_2 = u_2(f + (c + d)g).$$

Note that in particular, $c \in \mathbb{R} \setminus \mathbb{Q}$.

We will construct another totally ergodic system $(X', \mathcal{B}', \mu', T')$ for which $([p_1(n)])_n, ([p_2(n)])_n$ are not totally jointly ergodic. We have the following cases.

Case 1. f is a multiple of g . Here, we can rewrite the polynomials as

$$p_1 = u_1 c' g', \quad p_2 = u_2 (c' + d') g'$$

for some $g' \in \mathbb{Q}[x]$, $g'(0) = 0$, $g' \not\equiv 0$, $c' \in \mathbb{R} \setminus \mathbb{Q}$, $d' \in \mathbb{Q}_*$. Let t_1, t_2 be irrational numbers such that the following conditions are satisfied:

- $t_1 u_1 c' + t_2 u_2 (c' + d') = c' = s c' + t$ (for $s = 1$ and $t = 0$);
- $\frac{t_2 d'}{u_1 c'} \notin \mathbb{Z}$;
- t_1, t_2, c' and 1 are \mathbb{Q} -independent.²³

By Lemma 3.5, there exists a totally ergodic system $(X', \mathcal{B}', \mu', T')$ such that $S(T') = \text{span}_{\mathbb{Z}}\{t_1, t_2, c', 1\}$. Then, (p_1, p_2) is of Type-B for $(X', \mathcal{B}', \mu', T')$. Since $s = 1 \in \mathbb{Z}$ and $\frac{t - t_2 u_2 d'}{u_1 u_2 c'} = -\frac{t_2 d'}{u_1 c'} \notin \mathbb{Z}$, we have, by Proposition 5.7 (ii) and Lemma 5.3, that there exists W_0 depending only on p_1, p_2, t_1 and t_2 such that $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$ -jointly ergodic for $(X', \mathcal{B}', \mu', T')$ for all $W \geq W_0$. So $([p_1(n)])_n, ([p_2(n)])_n$ are not totally jointly ergodic for $(X', \mathcal{B}', \mu', T')$.

Case 2. f is not a multiple of g . Here we have two further sub-cases.

Case 2.1. $|u_1 u_2| > 1$.

Since $|u_1 u_2| > 1$, there exists $a \in \mathbb{Q}$ such that $\frac{ad}{u_1} \notin \mathbb{Z}_*$ and $au_2 d \in \mathbb{Z}$ (e.g., one can simply take $a = \frac{1}{u_2 d}$). By Lemma 3.5, there exists a totally ergodic system $(X', \mathcal{B}', \mu', T')$ such that $S(T') = \text{span}_{\mathbb{Z}}\{c' := \frac{c}{M}, 1\}$, where $M \in \mathbb{N}$ will be chosen later. We may then rewrite

$$p_1 = u_1 (f + c' g'), \quad p_2 = u_2 (f + (c' + d') g'),$$

where $g' = Mg$ and $d' = \frac{d}{M}$. Let $a' = aM$, $t_1 = -\frac{u_2}{u_1} a' c' = -\frac{aM u_2}{u_1} c'$ and $t_2 = a' c' = aM c'$. If we pick M so that $\frac{aM}{u_1} \in \mathbb{Z}$, then $t_1, t_2 \in S(T')$. Note that $t_1 u_1 + t_2 u_2 = 0$. On the other hand, note that $\frac{a' d'}{u_1} = \frac{ad}{u_1} \notin \mathbb{Z}_*$ and $a' u_2 d' = au_2 d \in \mathbb{Z}$. So, by Proposition 5.5 (iii) and Lemma 5.3, there exists W_0 depending only on p_1, p_2, t_1 and t_2 such that $([p_1(n)])_n, ([p_2(n)])_n$ are not $W!$ -jointly ergodic for $(X', \mathcal{B}', \mu', T')$ for all $W \geq W_0$. So $([p_1(n)])_n, ([p_2(n)])_n$ are not totally jointly ergodic for $(X', \mathcal{B}', \mu', T')$.

Case 2.2. $|u_1 u_2| = 1$.

In this case, we may assume without loss of generality that $u_1 = 1$ and $u_2 = \pm 1$. Since condition (ii)' fails, we have that $dg(r) \notin \mathbb{Z}$ for some $r \in \mathbb{Z}$. Let $W \in \mathbb{N}$ be such that $f(Wn + r) - f(r), dg(Wn + r) - dg(r) \in \mathbb{Z}$ for all $n \in \mathbb{N}$. By Lemma 3.5, there exists a totally ergodic system $(X', \mathcal{B}', \mu', T')$ such that $c/d \in S(T')$.

²³This can be easily seen as follows: We want to pick $t_2 \notin \mathbb{Q}$ such that $t_1 u_1 c' + t_2 u_2 (c' + d') = c'$, $d' t_2 \neq k c' u_1$ for all $k \in \mathbb{Z}$, and $s_1 t_1 + s_2 t_2 + s_3 c' \neq s_0$ for all $s_0, s_1, s_2, s_3 \in \mathbb{Q}$ not all 0. This is equivalent in picking t_2 irrational with $(s_2 u_1 c' - s_1 u_2 (c' + d')) t_2 \neq s_0 u_1 c' - s_1 c' - s_3 u_1 (c')^2$ for all $(s_0, s_1, s_2, s_3) \in \mathbb{Q}^4 \setminus \{(0, 0, 0, 0)\}$, and $d' t_2 \neq k c' u_1$ for all $k \in \mathbb{Z}$. It is obvious that there are uncountably many such choices.

We first consider the case $u_2 = 1$. Let $t_2 = -t_1 = c/d$. Then, (24) is equal to

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e\left(-t_1\{f(m) + cg(m)\} - t_2\{f(m) + (c+d)g(m)\}\right. \\
(25) \quad & \left. + \{(t_1 + t_2)(f(m) + cg(m)) + t_2 dg(m)\}\right) \\
& = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e\left(\frac{c}{d}(\{f(r) + cg(m)\} - \{f(r) + dg(r) + cg(m)\}) + \{cg(m)\}\right).
\end{aligned}$$

By Weyl's criterion, we have that $(\{cg(Wn+r)\})_n$ is equidistributed on \mathbb{T} .

If $\{f(r)\} + \{dg(r)\} < 1$, then (25) is equal to

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e\left(\frac{c}{d}(\mathbf{1}_{\{cg(m)\} \in (1-\{f(r)\}-\{dg(r)\}, 1-\{f(r)\})} - \{dg(r)\}) + \{cg(m)\}\right) \\
& = \int_0^1 e\left(\frac{c}{d}(\mathbf{1}_{x \in (1-\{f(r)\}-\{dg(r)\}, 1-\{f(r)\})} - \{dg(r)\}) + x\right) dx \\
& = \int_{1-\{f(r)\}-\{dg(r)\}}^{1-\{f(r)\}} e\left(\frac{c}{d}(1 - \{dg(r)\}) + x\right) dx + \int_{1-\{f(r)\}}^{2-\{f(r)\}-\{dg(r)\}} e\left(-\frac{c}{d}\{dg(r)\} + x\right) dx \\
& = \frac{e(\frac{c}{d}) - 1}{2\pi i} e\left(-\{f(r)\} - \frac{c}{d}\{dg(r)\}\right) (1 - e(-\{dg(r)\})).
\end{aligned}$$

If $\{f(r)\} + \{dg(r)\} \geq 1$, then (25) is equal to

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e\left(\frac{c}{d}(\mathbf{1}_{\{cg(m)\} \notin (1-\{f(r)\}, 2-\{f(r)\}-\{dg(r)\})} - \{dg(r)\}) + \{cg(m)\}\right) \\
& = \int_0^1 e\left(\frac{c}{d}(\mathbf{1}_{x \notin (1-\{f(r)\}, 2-\{f(r)\}-\{dg(r)\})} - \{dg(r)\}) + x\right) dx \\
& = \left(\int_0^{1-\{f(r)\}} + \int_{2-\{f(r)\}-\{dg(r)\}}^1\right) e\left(\frac{c}{d}(1 - \{dg(r)\}) + x\right) dx \\
& \quad + \int_{1-\{f(r)\}}^{2-\{f(r)\}-\{dg(r)\}} e\left(-\frac{c}{d}\{dg(r)\} + x\right) dx \\
& = \int_{2-\{f(r)\}-\{dg(r)\}}^{2-\{f(r)\}} e\left(\frac{c}{d}(1 - \{dg(r)\}) + x\right) dx + \int_{1-\{f(r)\}}^{2-\{f(r)\}-\{dg(r)\}} e\left(-\frac{c}{d}\{dg(r)\} + x\right) dx \\
& = \frac{e(\frac{c}{d}) - 1}{2\pi i} e\left(-\{f(r)\} - \frac{c}{d}\{dg(r)\}\right) (1 - e(-\{dg(r)\})).
\end{aligned}$$

In both cases above, since $dg(r) \notin \mathbb{Z}$, the averages are nonzero. So $([p_1(n)])_n, ([p_2(n)])_n$ are not totally jointly ergodic for $(X', \mathcal{B}', \mu', T')$.

Finally, we consider the case $u_2 = -1$. Let $t_2 = t_1 = c/d$. Using the fact that $[-x] + [x] = -1$ for all $x \in \mathbb{R} \setminus \mathbb{Z}$, we may invoke the previous computation to conclude that (24) is equal

to

$$\frac{-e(\frac{c}{d})(e(\frac{c}{d}) - 1)}{2\pi i} e\left(-\{f(r)\} - \frac{c}{d}\{dg(r)\}\right) (1 - e(-\{dg(r)\})),$$

i.e., it differs from the value of the first case by a the factor $-e(\frac{c}{d})$. So, we conclude that $([p_1(n)])_n, ([p_2(n)])_n$ are not totally jointly ergodic for $(X', \mathcal{B}', \mu', T')$.

6.2.2. Necessity. To prove the necessity part of Theorem B, we deal with the following two cases.

Condition (i) holds. Since p_1, p_2 are $\mathbb{R} \setminus \mathbb{Q}_*$ -independent, then, for any totally ergodic system (X, \mathcal{B}, μ, T) , p_1, p_2 are $S(T) \setminus \mathbb{Z}_*$ -independent and the result follows from Proposition 3.1.

Condition (ii)' holds. Assume without loss of generality that $u_1 = 1$. It suffices to show that for any totally ergodic system (X, \mathcal{B}, μ, T) , $t_1, t_2 \in S(T)$ not both in \mathbb{Z} , $W \in \mathbb{N}$, and $r \in \mathbb{Z}$, we have that (24) equals 0.

We first assume that $u_2 = 1$. Then, (24) equals to

(26)

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e\left(-t_1\{f(m) + cg(m)\} - t_2\{f(m) + (c+d)g(m)\}\right) \\ & \qquad \qquad \qquad + \{(t_1 + t_2)(f(m) + cg(m)) + t_2dg(m)\}) \\ & = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e\left(- (t_1 + t_2)\{f(m) + cg(m)\} + \{(t_1 + t_2)(f(m) + cg(m)) + t_2dg(m)\}\right). \end{aligned}$$

We first consider the case $t_1 + t_2 \in \mathbb{Q}$. Since X is totally ergodic, we have that $t_1 + t_2 \in \mathbb{Z}$. Since (24) remains unchanged if we replace t_1 by $t_1 + k$ for any $k \in \mathbb{Z}$, we may then assume without loss of generality that $t_1 + t_2 = 0$. Then, (26) equals to

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{\substack{m=Wn+r, \\ 1 \leq n \leq N}} e(t_2dg(m)),$$

which by Weyl's criterion converges to 0 if $t_2 \notin \mathbb{Q}$. Actually, the latter holds since if $t_2 \in \mathbb{Q}$, then $t_2 \in \mathbb{Z}$ by total ergodicity. So, we also have $t_1 \in \mathbb{Z}$, a contradiction.

We next consider the case $t_1 + t_2 \notin \mathbb{Q}$. Suppose that

$$(a + b(t_1 + t_2))f + ((a + b(t_1 + t_2))c + bt_2d)g = a(f + cg) + b((t_1 + t_2)(f + cg) + t_2dg) \in \mathbb{Q}[x]$$

for some $a, b \in \mathbb{Q}$. By Lemma 5.4, which can be used as f is not a multiple of g , we have that $a + b(t_1 + t_2) \in \mathbb{Q}$. This implies that $b = 0$, which in turn forces $a = 0$. Therefore, $(\{(f + cg)(Wn + r)\}, \{(t_1 + t_2)(f + cg) + t_2dg)(Wn + r)\})_n$ is equidistributed on \mathbb{T}^2 by Weyl's criterion. So, (26) equals to

$$\int_{\mathbb{T}^2} e(-(t_1 + t_2)x + y) dx dy = 0,$$

as was to be shown.

In the case where $u_2 = -1$, notice that for all $t_1, t_2 \in \mathbb{R}$ and $n \in \mathbb{N}$, we have that

$$e(t_1[p_1(n)] + t_2[p_2(n)]) = e(t_1[p_1(n)] - t_2[-p_2(n)]) \cdot e(-t_2).$$

We may use the previous case to conclude the proof, i.e., for any totally ergodic system (X, \mathcal{B}, μ, T) , $t_1, t_2 \in S(T)$ not both in \mathbb{Z} , and $W \in \mathbb{N}, r \in \mathbb{Z}$, we have that (24) equals 0.

7. APPENDIX: TWO INTERESTING EXAMPLES

In this appendix, we provide two examples illustrating how minor changes in the polynomial iterates can essentially affect their joint ergodicity properties.

The first one shows that, for some $a \in \mathbb{R}$, $([p_1(n)])_n, ([p_2(n)])_n$ can behave differently than $([p_1(n)])_n, ([p_2(n) + a])_n$.

Proposition 7.1. *There exist $p_1, p_2 \in \mathbb{R}[x]$ and $a \in \mathbb{R}$ such that $([p_1(n)])_n, ([p_2(n)])_n$ are totally jointly ergodic for all totally ergodic systems, and there exists a totally ergodic system (X, \mathcal{B}, μ, T) , such that for all $W \in \mathbb{N}$ and $r \in \mathbb{Z}$, $([p_1(Wn+r)])_n, ([p_2(Wn+r)+a])_n$ are not jointly ergodic for (X, \mathcal{B}, μ, T) .*

Proof. Let $W \in \mathbb{N}$, $r \in \mathbb{Z}$, $c \in \mathbb{R} \setminus \mathbb{Q}$, and

$$p_1(n) = n^2 + cn, \quad p_2(n) = n^2 + (c+1)n.$$

By Theorem B, $([p_1(n)])_n, ([p_2(n)])_n$ are totally jointly ergodic for all totally ergodic systems.

On the other hand, let $(X, \mathcal{B}, \mu, T) = (\mathbb{T}, \mathcal{B}, m, T)$, where m is the Haar measure on \mathbb{T} and $Tx = x + c \pmod{1}$ for all $x \in \mathbb{T}$. Then

(27)

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(-c[p_1(Wn+r)] + c\left[p_2(Wn+r) + \frac{1}{4}\right]\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(\frac{c}{4} + (c+1)\{c(Wn+r)\} - c\left\{c(Wn+r) + \frac{1}{4}\right\}\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(\mathbf{1}_{\{c(Wn+r)\} \in [0, \frac{3}{4}]} e(\{c(Wn+r)\}) + \mathbf{1}_{\{c(Wn+r)\} \in [\frac{3}{4}, 1]} e(c + \{c(Wn+r)\}) \right). \end{aligned}$$

Since $(c(Wn+r))_n$ is equidistributed on \mathbb{T} by Weyl's criterion, we have that (27) is equal to

$$\int_0^{\frac{3}{4}} e(t) dt + \int_{\frac{3}{4}}^1 e(c+t) dt = \frac{1+i}{2\pi i} (e(c) - 1) \neq 0.$$

So $([p_1(Wn+r)])_n, ([p_2(Wn+r) + \frac{1}{4}])_n$ are not jointly ergodic for (X, \mathcal{B}, μ, T) . \square

The second example provides two polynomial sequences which, for all $W \geq 2$, are $W!$ -jointly ergodic for every totally ergodic system, but are not jointly ergodic for some totally ergodic system.

Proposition 7.2. *There exist $p_1, p_2 \in \mathbb{R}[x]$ such that $([p_1(2n)])_n, ([p_2(2n)])_n$ are totally jointly ergodic for all totally ergodic systems, and there exists a totally ergodic system (X, \mathcal{B}, μ, T) , such that $([p_1(n)])_n, ([p_2(n)])_n$ are not jointly ergodic for (X, \mathcal{B}, μ, T) .*

Proof. Let $c \in \mathbb{R} \setminus \mathbb{Q}$, and

$$p_1(n) = n^3 + \frac{cn^2}{4}, \quad p_2(n) = n^3 + \frac{(c+1)n^2}{4}.$$

By Theorem B, $([p_1(2n)])_n, ([p_2(2n)])_n$ are totally jointly ergodic for all totally ergodic systems.

Let $(X, \mathcal{B}, \mu, T) = (\mathbb{T}, \mathcal{B}, m, T)$, where m is the Haar measure on \mathbb{T} and $Tx = x + c \pmod{1}$ for all $x \in \mathbb{T}$. We will show that $([p_1(n)])_n, ([p_2(n)])_n$ are not jointly ergodic for (X, \mathcal{B}, μ, T) . Since $([p_1(2n)])_n, ([p_2(2n)])_n$ are jointly ergodic for (X, \mathcal{B}, μ, T) , it suffices to show that $([p_1(2n+1)])_n, ([p_2(2n+1)])_n$ are not jointly ergodic for (X, \mathcal{B}, μ, T) . To this end, notice that

$$\begin{aligned} & \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left(-c[p_1(2n+1)] + c[p_2(2n+1)]\right) \\ &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N e\left((c+1)\left\{\frac{c(2n+1)^2}{4}\right\} - c\left\{\frac{c(2n+1)^2}{4} + \frac{1}{4}\right\}\right) \\ (28) \quad &= \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{n=1}^N \left(\mathbf{1}_{\left\{\frac{c(2n+1)^2}{4}\right\} \in [0, \frac{3}{4})} e\left(\left\{\frac{c(2n+1)^2}{4}\right\}\right) \right. \\ & \quad \left. + \mathbf{1}_{\left\{\frac{c(2n+1)^2}{4}\right\} \in [\frac{3}{4}, 1)} e\left(c + \left\{\frac{c(2n+1)^2}{4}\right\}\right) \right) e\left(-\frac{c}{4}\right). \end{aligned}$$

Since $c \notin \mathbb{Q}$, $\left(\frac{c(2n+1)^2}{4}\right)_n$ is equidistributed on \mathbb{T} by Weyl's criterion. So, (28) is equal to

$$e\left(-\frac{c}{4}\right) \left(\int_0^{\frac{3}{4}} e(t) dt + \int_{\frac{3}{4}}^1 e(c+t) dt \right) = \frac{1+i}{2\pi i} e\left(-\frac{c}{4}\right) (e(c) - 1) \neq 0.$$

Thus $([p_1(n)])_n, ([p_2(n)])_n$ are not jointly ergodic for (X, \mathcal{B}, μ, T) . \square

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