TOPOLOGICAL DYNAMICS ON NETS

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Abstract. We introduce the notion of a topological dynamical system indexed by a directed partial semigroup, and we prove recurrent and multiple recurrent results for such systems indexed by an arbitrary directed partial semigroup, with respect to a coideal basis suitable for this semigroup, but otherwise arbitrary. Stronger results are obtained for the subclass of those semigroups possessing digital representation. Our theory includes the recurrent and multiple recurrent results proved by Furstenberg and Weiss for dynamical systems indexed by integers, or by finite non-empty subsets of integers, and the analogous results for dynamical systems indexed by words proved by Farmaki and Koutsogiannis.

1. Introduction

According to a classical theorem by Birkhoff, in [2],the originator of topological dynamical theory, if $T$is a continuous function from a compact space to itself, then there exists $x \in X$ such that $\lim_{k\in\mathbb{N}} T^{n_k}(x) = x$ for some sequence $(n_k)_{k\in\mathbb{N}}$ of integers.

Birkhoff's result was generalized by Furstenberg and Weiss in [13] in two directions. Firstly, defining the recurrent points of X to be the elements of X such that $T^{n_k}(x) = x$ for some sequence $(n_k)_{k\in\mathbb{N}}$ of integers, they located conditions under which a subset of X will have recurrent points; secondly, they established a multiple Birkhoff's type recurrence theorem. Furstenberg and Weiss ([13], [11]) extended these results to systems $(T^F)_{F\in[\mathbb{N}]^{<\infty}_{>0}}$, where $[\mathbb{N}]^{<\infty}_{>0}$ is the set of all finite non-empty subsets of $\mathbb{N}$, instead of the systems $(T^n)_{n\in[\mathbb{N}]}$ used earlier. Analogous recurrent and multiple recurrent results for systems indexed by words were proved by Furstenberg and Katznelson in [12], and, recently, by Farmaki and Koutsogiannis in [7].

In this paper we present a general topological dynamical theory, which includes and unifies the previous results. We prove recurrent and multiple recurrent results for systems indexed by an arbitrary directed partial semigroup, with respect to an arbitrary coideal basis suitable for this semigroup. Moreover, we obtain stronger results for the subclass of those semigroups possessing digital representation.

We will use the following notation.

Notation. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers. For a set X we denote by $[X]^{<\infty}$ the set of all finite subsets of X, by $[X]^{<\infty}_{>0}$ the set of all finite non-empty subsets of X and by $[X]^{\infty}$ the set of all infinite subsets of X.

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2. Coideal bases with the (D)-property

In this section we will introduce the (D)-property of a coideal basis on an infinite directed set $(\Lambda, \prec)$ and we will prove, in Theorem 2.10 below, that every net $(x_\lambda)_{\lambda\in\Lambda}$ in a compact metric space has a convergent subnet of the form $(x_\lambda)_{\lambda\in A}$, where A is an element of an arbitrary coideal basis $\mathcal{B}$ on Λ with the (D)-property. Moreover, we will locate A to be a subset of a given element B of the coideal basis $\mathcal{B}$. This result will be the starting point in order to prove later recurrence results for topological systems of continuous maps from a compact metric space into itself indexed by an infinite directed set with respect to a coideal basis with the (D)-property.

The notion of a coideal on the set of natural numbers appears in [15], [5], [17] and elsewhere. This notion extended in [9] from the set of natural numbers to an arbitrary infinite directed set as follows:

Definition 2.1. Let Λ be a non-empty infinite set and $\prec$ a relation on Λ satisfying the following conditions:

(i) If $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$, then $\lambda_1 \neq \lambda_2$.
(ii) If $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_2$ and $\lambda_2 \prec \lambda_3$, then $\lambda_1 \prec \lambda_3$.
(iii) For every $\lambda_1, \lambda_2 \in \lambda$ there exists $\lambda_3 \in \Lambda$ such that $\lambda_1 \prec \lambda_3$ and $\lambda_2 \prec \lambda_3$.

Then $(\Lambda, \prec)$ is a **directed** set.

Definition 2.2. Let $(\Lambda, \prec)$ be an infinite directed set. A subset $\mathcal{H}$ of $[\Lambda]^\infty$ is a **coideal** on $(X, \prec)$ if it satisfies the following three properties:

(i) For every $A \in \mathcal{H}$ and $\lambda_1 \in \Lambda$ there exists $\lambda_2 \in A$ such that $\lambda_1 \prec \lambda_2$.
(ii) If $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$ or $B \in \mathcal{H}$.
(iii) If $A \in \mathcal{H}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{H}$.

Notation. Let $(\Lambda, \prec)$ be an infinite directed set and let $A \subseteq \Lambda$ and $\lambda \in \Lambda$. Then,

$$A - \lambda = \{z \in A : \lambda \prec z\}.$$

Remarks 2.3. (i) A nonprincipal ultraflter on $\mathbb{N}$ is a coideal on $\mathbb{N}$, which is also closed under the finite intersections. So, $[\mathbb{N}]^\infty$ is a coideal on $\mathbb{N}$ which is not an ultrafilter.

(ii) Let $(\Lambda, \prec)$ be an infinite directed set. An ultrafter $\mathcal{U}$ on Λ such that $A - \lambda \neq \emptyset$, for every $A \in \mathcal{U}$ and $\lambda \in \Lambda$, is a coideal on $(\Lambda, \prec)$, and moreover every union of such ultrafilters on Λ is a coideal on $(\Lambda, \prec)$.

We will give now the notion of a coideal basis on an infinite directed set, which also defined in [9].

Definition 2.4. Let $(\Lambda, \prec)$ be an infinite directed set. A subset $\mathcal{B}$ of $[\Lambda]^\infty$ is a **coideal basis** on $(\Lambda, \prec)$ if it satisfies the following two properties:

(i) For every $A \in \mathcal{B}$ and $\lambda_1 \in \Lambda$ there exists $\lambda_2 \in A$ such that $\lambda_1 \prec \lambda_2$.
(ii) If $A \cup B \in \mathcal{B}$, then there exists $C \in \mathcal{B}$ such that either $C \subseteq A$ or $C \subseteq B$.

Obviously, a coideal on $(\Lambda, \prec)$ is a coideal basis on $(\Lambda, \prec)$. The connection between coideals and coideal bases is given in the following proposition.

Proposition 2.5. *Let $(\Lambda, \prec)$ be an infinite directed set. A family $\mathcal{H} \subseteq [\Lambda]^\infty$ is a coideal on $(\Lambda, \prec)$ if and only if there exists a coideal basis $\mathcal{B} \subseteq [\Lambda]^\infty$ such that*

$$\mathcal{H} = \mathcal{L}_{\mathcal{B}} = \{A \subseteq X : \text{ there exists } B \in \mathcal{B} \text{ with } B \subseteq A\}.$$

Remarks 2.6. Let $(\Lambda,\prec)$ be an infinite directed set and $\mathcal{B}$ a coideal basis on $(\Lambda,\prec)$.

(i) Every element A of $\mathcal{B}$ is an infinite directed set.

(ii) If $(x_\lambda)_{\lambda\in\Lambda}$ is a net in a set X, then $(x_\lambda)_{\lambda\in A}$ is a subnet of $(x_\lambda)_{\lambda\in\Lambda}$ for every $A\in\mathcal{B}$.

We will define now the (P)-property of a coideal basis on an infinite directed set, extending the analogous notion defined in [15], [5], [17] for coideals on the set of natural numbers. An ultrafilter on the set of natural numbers with the (P)-property is called P-point ultrafilter.

Definition 2.7. Let $(\Lambda,\prec)$ be an infinite directed set. A coideal basis $\mathcal{B}\subseteq[\Lambda]^\infty$ on $(\Lambda,\prec)$ has the **(P)-property** if for every sequence $(A_n)_{n\in\mathbb{N}}$, with $A_n\in\mathcal{B}$ and $A_1\supseteq A_2\supseteq\ldots$, there exists $A\in\mathcal{B}$ such that $A\setminus A_n$ is a finite set for every $n\in\mathbb{N}$.

We will introduce now a weaker property than the (P)-property of a coideal basis on an infinite directed set, which we call (D)-property.

Definition 2.8. Let $(\Lambda,\prec)$ be an infinite directed set. A coideal basis $\mathcal{B}\subseteq[\Lambda]^\infty$ on $(\Lambda,\prec)$ has the **(D)-property** if for every sequence $(A_n)_{n\in\mathbb{N}}$, with $A_n\in\mathcal{B}$ and $A_1\supseteq A_2\supseteq\ldots$, there exists $A\in\mathcal{B}$, such that for every $n\in\mathbb{N}$ there exists $k_n\in\mathbb{N}\cup\{0\}$ satisfying

$$k_n=\max\{k\in\mathbb{N}: \text{there exist } \lambda_1,\ldots,\lambda_k\in A\setminus A_n \text{ with } \lambda_1\prec\cdots\prec\lambda_k\}.$$

Examples 2.9. **(1)** The set $[\mathbb{N}]^\infty$ is a coideal on $\mathbb{N}$ with the usual order, according to the pigeon-hole principle, and obviously has the (P)-property and consequently the (D)-property.

(2) Let $[\mathbb{N}]^{<\infty}_{>0}$ be the set of all the finite non-empty subsets of $\mathbb{N}$. For $F_1,F_2\in[\mathbb{N}]^{<\infty}_{>0}$ we define $F_1\prec F_2$ if $\max F_1<\min F_2$. Then $([\mathbb{N}]^{<\infty}_{>0},\prec)$ is an infinite directed set.

For a sequence $(F_n)_{n\in\mathbb{N}}\subseteq[\mathbb{N}]^{<\infty}_{>0}$ such that $F_n\prec F_{n+1}$ for every $n\in\mathbb{N}$ we set $FU((F_n)_{n\in\mathbb{N}})=\{\bigcup_{i\in\alpha}F_i:\ \alpha\in[\mathbb{N}]^{<\infty}_{>0}\}$. The family

$$\mathcal{B}=\{FU((F_n)_{n\in\mathbb{N}}):(F_n)_{n\in\mathbb{N}}\subseteq[\mathbb{N}]^{<\infty}_{>0} \text{ with } F_1\prec F_2\prec\ldots\}$$

is a coideal basis on $([\mathbb{N}]^{<\infty}_{>0},\prec)$, according to the fundamental theorem of Hindman ([14]).

This coideal basis has not the (P)-property, but has the (D)-property. Indeed, let a sequence $(A_k)_{k\in\mathbb{N}}$, with $A_k\in\mathcal{B}$ and $A_1\supseteq A_2\supseteq\ldots$. If $A_k=FU((F^k_n)_{n\in\mathbb{N}})$, where $(F^k_n)_{n\in\mathbb{N}}\subseteq[\mathbb{N}]^{<\infty}_{>0}$ with $F^k_1\prec F^k_2\prec\ldots$ for every $k\in\mathbb{N}$, then we set $A=FU((F^k_k)_{k\in\mathbb{N}})$. Then $A\in\mathcal{B}$ and for every $k\in\mathbb{N}$

$$k-1=\max\{n\in\mathbb{N}: \text{there exist } F_1\prec\cdots\prec F_n\in A\setminus A_k \text{ with } F_1\prec\cdots\prec F_n\}.$$

(3) Let $\Sigma=\{\alpha_1,\alpha_2,\ldots\}$ be an infinite countable alphabet and $\vec{k}=(k_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ an increasing sequence. The set of **ω-located words** over Σ dominated by $\vec{k}$ is

$$L(\Sigma,\vec{k})=\{w=w_{n_1}\ldots w_{n_l}: l\in\mathbb{N}, n_1<\ldots<n_l\in\mathbb{N},\ w_{n_i}\in\{\alpha_1,\ldots,\alpha_{k_{n_i}}\} \text{ for all } 1\leq i\leq l\}.$$

Let $\upsilon\notin\Sigma$ be a variable. The set of **variable ω-located words** over Σ dominated by the sequence $\vec{k}$ is:

$$L(\Sigma,\vec{k};\upsilon)=\{w=w_{n_1}\ldots w_{n_l}: l\in\mathbb{N},\ n_1<\ldots<n_l\in\mathbb{N},\ w_{n_i}\in\{\upsilon,\alpha_1,\ldots,\alpha_{k_{n_i}}\} \text{ for all } 1\leq i\leq l \text{ and there exists } 1\leq i\leq l \text{ with } w_{n_i}=\upsilon\}.$$

Let $L(\Sigma\cup\{\upsilon\},\vec{k})=L(\Sigma,\vec{k})\cup L(\Sigma,\vec{k};\upsilon)$.

If $w=w_{n_1}\ldots w_{n_l}\in L(\Sigma\cup\{\upsilon\},\vec{k})$, then the set $dom(w)=\{n_1,\ldots,n_l\}$ is the *domain* of w. For $w,u\in L(\Sigma\cup\{\upsilon\},\vec{k})$ we define $w\prec u$ if $\max dom(w)<\min dom(u)$. Then $(L(\Sigma\cup\{\upsilon\},\vec{k}),\prec)$, $(L(\Sigma,\vec{k}),\prec)$ and $(L(\Sigma,\vec{k};\upsilon),\prec)$ are infinite directed sets.

For $w=w_{n_1}\ldots w_{n_l}\in L(\Sigma,\vec{k};\upsilon)$ and $p\in\mathbb{N}\cup\{0\}$ we set $w(0)=w$ and, for $p\in\mathbb{N}$,

$$w(p) = u_{n_1} \dots u_{n_l} \in L(\Sigma, \vec{k})$$

where, for $1 \le i \le l$, $u_{n_i} = w_{n_i}$ if $w_{n_i} \in \Sigma$, $u_{n_i} = \alpha_p$ if $w_{n_i} = \upsilon$ and $p \le k_{n_i}$ and finally $u_{n_i} = \alpha_{k_{n_i}}$ if $w_{n_i} = \upsilon$ and $p > k_{n_i}$. Let

$$L^\omega(\Sigma, \vec{k}; \upsilon) = \{(w_n)_{n\in\mathbb{N}} \subseteq L(\Sigma, \vec{k}; \upsilon) : w_n \prec w_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

We will define now the extracted (variable) ω-located words of a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)$. An **extracted variable ω-located words** of $\vec{w}$ has the form

$$w_{n_1}(p_1) \star \dots \star w_{n_\lambda}(p_\lambda) \in L(\Sigma, \vec{k}; \upsilon),$$

where $\lambda \in \mathbb{N}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$ and $p_1, \dots, p_\lambda \in \mathbb{N} \cup \{0\}$ with $0 \le p_i \le k_{n_i}$ for every $1 \le i \le \lambda$ and $0 \in \{p_1, \dots, p_\lambda\}$. The set of all the extracted variable ω-located words of $\vec{w}$ is denoted by $EV(\vec{w})$.

An **the extracted ω-located words** of $\vec{w}$ has the form

$$w_{n_1}(p_1) \star \dots \star w_{n_\lambda}(p_\lambda) \in L(\Sigma, \vec{k}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$ and $p_1, \dots, p_\lambda \in \mathbb{N}$ with $1 \le p_i \le k_{n_i}$ for every $1 \le i \le \lambda$. The set of all the extracted ω-located words of $\vec{w}$ is denoted by $E(\vec{w})$. The families

$$\mathcal{B} = \{E(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)\} \text{ and}$$
$$\mathcal{B}_1 = \{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)\}$$

are coideal bases on $(L(\Sigma, \vec{k}), \prec)$ and $(L(\Sigma, \vec{k}; \upsilon), \prec)$ respectively, according to a fundamental partition theorem of Carlson proved in [4] and [6] and in [1] for the particular case of a finite alphabet.

These coideal bases have not the (P)-property, but they have the (D)-property. Indeed, let a sequence $(A_k)_{k\in\mathbb{N}}$, with $A_k = E(\vec{w_k})$, where $\vec{w_k} = (w_n^k)_{n\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)$, and $A_1 \supseteq A_2 \supseteq \dots$. Let $\vec{w} = (w_k^k)_{k\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)$. We set $A = E(\vec{w})$. Then $A \in \mathcal{B}$. Moreover, for every $k \in \mathbb{N}$,

$$k - 1 = \max\{n \in \mathbb{N} : \text{there exist } w_1, \dots, w_n \in A \setminus A_k \text{ with } w_1 \prec \dots \prec w_n\}.$$

Hence, $\mathcal{B}$ has the (D)-property. Analogously, can be proved that $\mathcal{B}_1$ has the (D)-property.

(4) In the final section we will give more examples of coideal bases on directed sets with the (D)-property, using Example 3.

After the definition of the coideal bases on directed sets with the (D)-property, we can state the main theorem of this section. It is well known that every net $(x_\lambda)_{\lambda\in\Lambda}$ in a compact metric space has a convergent subnet. We will prove, in the following theorem, that this subnet can have the form $(x_\lambda)_{\lambda\in A}$, where A is an element of an arbitrary coideal basis $\mathcal{B}$ on Λ with the (D)-property, and moreover A can be a subset of a given element B of $\mathcal{B}$.

Let $(\Lambda, \prec)$ be an infinite directed set and $(x_\lambda)_{\lambda\in\Lambda} \subseteq X$ be a net in a topological space X. For $x_0 \in X$, we write

$$\lim_{\lambda\in\Lambda} x_\lambda = x_0,$$

if $(x_\lambda)_{\lambda\in\Lambda}$ converges to x_0, i.e. if for any neighborhood V of x_0, there exists $\lambda_0 \equiv \lambda_0(V) \in \Lambda$ such that $x_\lambda \in V$ for every $\lambda \in \Lambda$ with $\lambda_0 \prec \lambda$.

Analogously, we write for an element A of a coideal basis $\mathcal{B}$ on $(\Lambda, \prec)$ and $x_0 \in X$,

$$\lim_{\lambda\in A} x_\lambda = x_0,$$

if the net $(x_\lambda)_{\lambda\in A}$ converges to x_0, i.e. if for any neighborhood V of x_0, there exists $\lambda_0 \equiv \lambda_0(V) \in A$ such that $x_\lambda \in V$ for every $\lambda \in A$ with $\lambda_0 \prec \lambda$.

Theorem 2.10. *Let (X,d) be a compact metric space, $(\Lambda,\prec)$ an infinite directed set and let $(x_\lambda)_{\lambda\in\Lambda}\subseteq X$ a net in X. For every coideal basis $\mathcal{B}$ on $(\Lambda,\prec)$ with the (D)-property and every $B\in\mathcal{B}$ there exists $A\in\mathcal{B}$ with $A\subseteq B$ such that the subnet $(x_\lambda)_{\lambda\in A}$ of $(x_\lambda)_{\lambda\in\Lambda}$ to converge to some element of X.*

Proof. Let $\mathcal{B}$ be a coideal basis on $(\Lambda,\prec)$ with the (D)-property and $B\in\mathcal{B}$. We set $\widehat{B}(x,\epsilon)=\{y\in X: d(x,y)\le\epsilon\}$ for every $x\in X$ and $\epsilon>0$. Since (X,d) is a compact metric space, we have that $X=\bigcup_{i=1}^{m_1}\widehat{B}(x_i^1,\frac{1}{2})$ for some $x_1^1,\dots,x_{m_1}^1\in X$.

Let $A_1=B$. Since $A_1=\bigcup_{i=1}^{m_1}C_i$, where $C_i=\{\lambda\in A_1: x_\lambda\in\widehat{B}(x_i^1,\frac{1}{2})\}$, and $\mathcal{B}$ is a coideal basis, there exist $A_2\in\mathcal{B}$, $A_2\subseteq A_1$ and $1\le i_1\le m_1$ such that $A_2\subseteq C_{i_1}$ and consequently $\{x_\lambda:\lambda\in A_2\}\subseteq\widehat{B}(x_{i_1}^1,\frac{1}{2})$. We continue analogously. Since $\widehat{B}(x_{i_1}^1,\frac{1}{2})$ is a compact space, there exist $x_1^2,\dots,x_{m_2}^2\in X$, such that $\widehat{B}(x_{i_1}^1,\frac{1}{2})\subseteq\bigcup_{i=1}^{m_2}\widehat{B}(x_i^2,\frac{1}{4})$, and consequently there exist $A_3\in\mathcal{B}$, $A_3\subseteq A_2$ and $1\le i_2\le m_2$ such that $\{x_\lambda:\lambda\in A_3\}\subseteq\widehat{B}(x_{i_1}^1,\frac{1}{2})\cap\widehat{B}(x_{i_2}^2,\frac{1}{4})$.

Inductively, we construct a sequence $(A_n)_{n\in\mathbb{N}}$, with $A_n\in\mathcal{B}$ and $A_1\supseteq A_2\supseteq\dots$, and also closed balls $\widehat{B}(x_{i_n}^n,\frac{1}{2^n})$, for $n\in\mathbb{N}$, such that

$$\{x_\lambda:\lambda\in A_{n+1}\}\subseteq\bigcap_{j=1}^{n}\widehat{B}(x_{i_j}^j,\tfrac{1}{2^j})\text{ for every }n\in\mathbb{N}.$$

Let $\{x_0\}=\bigcap_{n\in\mathbb{N}}\widehat{B}(x_{i_n}^n,\frac{1}{2^n})$. Since the coideal basis $\mathcal{B}$ has the (D)-property there exists $C\in\mathcal{B}$, such that for every $n\in\mathbb{N}$ there exists $k_n\in\mathbb{N}\cup\{0\}$ such that

$$k_n=\max\{k\in\mathbb{N}:\text{there exist }\lambda_1,\dots,\lambda_k\in C\setminus A_n\text{ with }\lambda_1\prec\dots\prec\lambda_k\}.$$

This implies that $C\setminus A_1$ does not contain an element of $\mathcal{B}$. So, since $C\in\mathcal{B}$ and $C=(C\setminus A_1)\cup(C\cap A_1)$, there exists $A\in\mathcal{B}$, $A\subseteq C\cap A_1$. Then $A\subseteq C$, $A\in\mathcal{B}$, $A\subseteq B=A_1$ and for every $n\in\mathbb{N}$, $n>1$ there exists $q_n\in\mathbb{N}\cup\{0\}$ such that

$$q_n=\max\{k\in\mathbb{N}:\text{there exist }\lambda_1,\dots,\lambda_k\in A\setminus A_n\text{ with }\lambda_1\prec\dots\prec\lambda_k\}.$$

We will prove that $\lim_{\lambda\in A}x_\lambda=x_0$. Indeed, for $\varepsilon>0$ pick $n_0\in\mathbb{N}$, such that $1/2^{n_0}<\varepsilon$. Then $d(x_\lambda,x_0)\le 1/2^{n_0}<\varepsilon$ for every $\lambda\in A_{n_0+1}$. Let $\lambda_1,\dots,\lambda_{q_{n_0+1}}\in A\setminus A_{n_0+1}$ with $\lambda_1\prec\dots\prec\lambda_{q_{n_0+1}}$. Since does not exist $\lambda\in A\setminus A_{n_0+1}$ with $\lambda_{q_{n_0+1}}\prec\lambda$ and $A\in\mathcal{B}$, there exists $\lambda_0\in A\cap A_{n_0+1}$ such that $\lambda_{q_{n_0+1}}\prec\lambda_0$. Hence, for every $\lambda\in A$ with $\lambda_0\prec\lambda$ we have that $\lambda\in A_{n_0+1}$ and consequently that $d(x_\lambda,x_0)\le 1/2^{n_0}<\varepsilon$. This finishes the proof. □

The particular case of Theorem 2.10 for the directed set $([\mathbb{N}]^{<\infty}_{>0},\prec)$ and the coideal basis $\mathcal{B}$ referred in Example 2.9,(2) was proved by Furstenberg and Weiss in [13]. Also, the particular case of Theorem 2.10 for the directed set $(L(\Sigma,\vec{k};\upsilon),\prec)$ of ω-located words and the coideal basis $\mathcal{B}$ referred in Example 2.9,(3) was proved by Farmaki and Koutsogiannis in [7] and in case the alphabet is finite by Furstenberg and Katznelson in [12].

3. Topological dynamical systems indexed by a directed partial semigroup and recurrent points

We will introduce the notion of a directed partial semigroup, consequently the suitable coideal bases on a directed partial semigroup and the topological dynamical systems indexed by a directed partial semigroup. Our aim is to prove recurrent results for topological dynamical systems indexed by a directed partial semigroup with respect to a suitable coideal basis for this semigroup extending fundamental recurrent results of Birkhoff ([2]), Furstenberg-Weiss ([13]) and Farmaki-Koutsogiannis ([7]). Lets start with the necessary definitions.

Definition 3.1. Let $(\Lambda, \prec)$ be an infinite directed set and let for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$ is defined a unique element $\lambda_1 * \lambda_2 \in \Lambda$. If for every $\lambda_1, \lambda_2, \lambda_3 \in \Lambda$ with $\lambda_1 \prec \lambda_2 \prec \lambda_3$ hold $\lambda_1 \prec \lambda_2 * \lambda_3$, $\lambda_1 * \lambda_2 \prec \lambda_3$ and $(\lambda_1 * \lambda_2) * \lambda_3 = \lambda_1 * (\lambda_2 * \lambda_3)$, then $(\Lambda, \prec, *)$ is called a **directed partial semigroup**.

Consequently, we will define the suitable coideal bases for a directed partial semigroup.

Definition 3.2. Let $(\Lambda, \prec, *)$ be a directed partial semigroup. A coideal basis $\mathcal{B}$ on $(\Lambda, \prec)$ is **suitable** for $(\Lambda, \prec, *)$ if every $B \in B$ has the property that $\lambda_1 * \lambda_2 \in B$ for every $\lambda_1, \lambda_2 \in B$ with $\lambda_1 \prec \lambda_2$.

Obviously, if a coideal basis $\mathcal{B}$ is suitable for the directed partial semigroup $(\Lambda, \prec, *)$, then $(B, \prec, *)$ is also a directed partial semigroup for every $B \in B$.

We will define now the central notion of this section, the notion of a topological dynamical system indexed by a directed partial semigroup.

Definition 3.3. Let $(\Lambda, \prec, *)$ be a directed partial semigroup. A family $\{T^\lambda\}_{\lambda \in \Lambda}$ of continuous functions from a compact metric space X into itself is a Λ-**topological dynamical system** of X if $T^{\lambda_1} \circ T^{\lambda_2} = T^{\lambda_1 * \lambda_2}$, for every $\lambda_1, \lambda_2 \in \Lambda$ with $\lambda_1 \prec \lambda_2$.

Obviously, if $\mathcal{B}$ is a suitable coideal basis for $(\Lambda, \prec, *)$ and $B \in B$, then the family $\{T^\lambda\}_{\lambda \in B}$ is also a topological dynamical system of X.

Examples 3.4. Let X be a compact metric space.
(1) Let $T : X \to X$ be a continuous map. Then $\{T^n\}_{n \in \mathbb{N}}$ is a $\mathbb{N}$-topological dynamical system of X.
(2) According to Example 2.9 (2), $([\mathbb{N}]^{<\infty}_{>0}, \prec)$ is an infinite directed set. So, $([\mathbb{N}]^{<\infty}_{>0}, \prec, \cup)$ is a directed partial semigroup and $\mathcal{B}$ a suitable coideal basis for this semigroup. For each $n \in \mathbb{N}$, let $T_n : X \to X$ be a continuous map. For $F = \{n_1 < \dots < n_k\} \in [\mathbb{N}]^{<\infty}_{>0}$ we set $T^F = T_{n_1} \circ \ldots \circ T_{n_k}$. Then $\{T^F\}_{F \in [\mathbb{N}]^{<\infty}_{>0}}$ is a $[\mathbb{N}]^{<\infty}_{>0}$-topological dynamical system of X. In particular, we can replace T_n with T^n, for every $n \in \mathbb{N}$, where $T : X \to X$ is a given continuous map.
(3) Let $\Sigma = \{\alpha_1, \alpha_2, \ldots\} \subseteq \mathbb{N}$ be an infinite countable alphabet and $\vec{k} = (k_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence. According to Example 2.9 (3), $(L(\Sigma, \vec{k}), \prec, \star)$ is a directed partial semigroup and $\mathcal{B}$ is a suitable coideal basis for this semigroup. Let $\{T_n\}_{n \in \mathbb{N}}$ be a sequence of continuous maps from X into itself and $(l_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$. For $w = w_{n_1} \ldots w_{n_\lambda} \in L(\Sigma, \vec{k})$ let

$$T^w = T_{n_1}^{l_{n_1} w_{n_1}} \circ \ldots \circ T_{n_\lambda}^{l_{n_\lambda} w_{n_\lambda}}.$$

Then $\{T^w\}_{w \in L(\Sigma, \vec{k})}$ is an $L(\Sigma, \vec{k})$-topological dynamical system of X.

Now, we will define the recurrent points of a topological dynamical system of a compact metric space, indexed by a directed partial semigroup, with respect to a suitable coideal basis on the semigroup. Consequently, using Theorem 2.10, we will prove the existence of such points in case the coideal basis has the (D)-property.

Definition 3.5. Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda \in \Lambda}$ a Λ-dynamical system of a compact metric space (X, d), $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$ and $B \in \mathcal{B}$. An element x_0 of X is called B-**recurrent** if

$$\lim_{\lambda \in A} T^\lambda(x_0) = x_0, \text{ for some } A \in \mathcal{B} \text{ with } A \subseteq B.$$

Theorem 3.6. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X, d), $\mathcal{B}$ a suitable for $(\Lambda, \prec, *)$ coideal basis with the (D)-property and $B \in \mathcal{B}$. Then X contains B-recurrent points.*

Proof. Let $x \in X$. According to Theorem 2.10, there exist $A \in \mathcal{B}$, $A \subseteq B$ and $x_0 \in X$ such that

$$\lim_{\lambda\in A} T^\lambda(x) = x_0.$$

Let $\varepsilon > 0$. Then, there exists $\lambda_0 \in A$ such that $d(T^\lambda(x), x_0) < \varepsilon/2$ for every $\lambda \in A$ with $\lambda_0 \prec \lambda$. We fix $\lambda_1 \in A$ with $\lambda_0 \prec \lambda_1$. Since T^{λ_1} is a continuous function on X, there exists $\delta > 0$ such that if $y \in X$ with $d(y, x_0) < \delta$, then $d(T^{\lambda_1}(y), T^{\lambda_1}(x_0)) < \varepsilon/2$. Since $\lim_{\lambda\in A} T^\lambda(x) = x_0$, there exists $\lambda_2 \in A$, $\lambda_1 \prec \lambda_2$ such that $d(T^{\lambda_2}(x), x_0) < \delta$. Then $d(T^{\lambda_1}(T^{\lambda_2}(x)), T^{\lambda_1}(x_0)) < \varepsilon/2$ and consequently $d(T^{\lambda_1*\lambda_2}(x), T^{\lambda_1}(x_0)) < \varepsilon/2$. Since $\lambda_1 * \lambda_2 \in A$ and $\lambda_0 \prec \lambda_1 * \lambda_2$, we have that $d(T^{\lambda_1*\lambda_2}(x), x_0) < \varepsilon/2$. It follows that $d(T^{\lambda_1}(x_0), x_0) < \varepsilon$. Hence, $\lim_{\lambda\in A} T^\lambda(x_0) = x_0$. □

The particular case of this theorem, where $\Lambda = \mathbb{N}$, $\mathcal{B} = [\mathbb{N}]^\infty$ and the system has the form $\{T^n\}_{n\in\mathbb{N}}$, where T is a continuous function from a compact metric space (X, d) to itself, is Birkhoff's recurrence theorem.

Corollary 3.7 (Birkhoff's theorem, [2]). *Let X be a compact metric space and $T : X \to X$ a continuous function. There exists $x_0 \in X$ and a sequence $(n_k)_{k\in\mathbb{N}}$ in $\mathbb{N}$ such that $\lim_{k\in\mathbb{N}} T^{n_k}(x_0) = x_0$.*

Our aim now is to locate recurrent points of a topological dynamical system indexed by a directed partial semigroup, with respect to a suitable coideal basis, in a given subset of the space. Firstly, we will look for almost recurrent points, as their class is wider than the class of recurrent points.

Definition 3.8. Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X, d), $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$ and $B \in \mathcal{B}$. An element x_0 of X is called B-**almost recurrent** if for every $\varepsilon > 0$ and $\lambda_0 \in \Lambda$, there exist $\lambda \in B$, $\lambda_0 \prec \lambda$ such that

$$d(T^\lambda(x_0), x_0) < \varepsilon.$$

A closed subset F of X is called B-**almost recurrent set** if for every $\varepsilon > 0$, for every $\lambda_0 \in \Lambda$ and every $x \in F$, there exist $y \in F$ and $\lambda \in B$, $\lambda_0 \prec \lambda$ such that

$$d(T^\lambda(y), x) < \varepsilon.$$

In the following example we point out a way to locate almost recurrent subsets of a compact metric space.

Example 3.9. Let (X, d) be a compact metric space, $(\Lambda, \prec, *)$ a directed partial semigroup, $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec)$ with the (D)-property and $B \in \mathcal{B}$. Let $F(X)$ be the set of all nonempty closed subsets of X endowed with the Hausdorff metric $\widetilde{d}$,where

$$\widetilde{d}(K, M) = \max\{\sup_{x\in K} d(x, M), \sup_{x\in M} d(x, K)\}.$$

Then $(F(X), \widetilde{d})$ is also a compact metric space. Let $\{T^\lambda\}_{\lambda\in\Lambda}$ be a Λ-dynamical system of (X, d). We define $\widetilde{T}^\lambda : F(X) \to F(X)$ with $\widetilde{T}^\lambda(K) = T^\lambda(K)$. Then $\{\widetilde{T}^\lambda\}_{\lambda\in\Lambda}$ is a Λ-dynamical system of $(F(X), \widetilde{d})$. According to Theorem 3.6, there exist $A \in \mathcal{B}$, $A \subseteq B$ and $K \in F(X)$ such that

$$\lim_{\lambda\in A} \widetilde{T}^\lambda(K) = K.$$

Then K is a B- recurrent element of $F(X)$ and K is a B- almost recurrent subset of X.

We will prove now that every almost recurrent subset of X contains almost recurrent elements of X.

Proposition 3.10. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X,d), $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$ and $B \in \mathcal{B}$. Every B- almost recurrent subset F of X contains B- almost recurrent elements of X.*

Proof. Let F be a B-almost recurrent subset of X. We fix $\varepsilon > 0$ and $\lambda_0 \in \Lambda$. Inductively, we will construct a sequence $(x_n)_{n\in\mathbb{N}} \subseteq F$, a sequence $(\lambda_n)_{n\in\mathbb{N}} \subseteq B$ with $\lambda_n \prec \lambda_{n+1}$ and a sequence $(\varepsilon_n)_{n\in\mathbb{N}}$ with $0 < \varepsilon_n < \varepsilon/2$, which for every $n \in \mathbb{N}$ satisfy $d(T^{\lambda_{n+1}}(x_{n+1}), x_n) < \varepsilon_{n+1}$ and $d(T^{\lambda_n}(x), x_{n-1}) < \varepsilon_n$ whenever $x \in X$ and $d(x, x_n) < \varepsilon_{n+1}$.
Indeed, since F is B-almost recurrent, for $x_0 \in F$ and $\varepsilon_1 = \varepsilon/2$ there exist $\lambda_1 \in B$ with $\lambda_0 \prec \lambda_1$ and $x_1 \in F$ such that $d(T^{\lambda_1}(x_1), x_0) < \varepsilon_1$.
Let there exist $x_0, x_1, \dots, x_n \in F$, $\lambda_1, \dots, \lambda_n \in B$ with $\lambda_1 \prec \dots \prec \lambda_n$ and $0 < \varepsilon_1, \varepsilon_2, \dots, \varepsilon_n < \varepsilon/2$ such that $d(T^{\lambda_i}(x_i), x_{i-1}) < \varepsilon_i$, for every $i = 1, \dots, n$.
Since T^{λ_n} is a continuous function, there exists $0 < \varepsilon_{n+1} \leqslant \varepsilon_n$ such that if $x \in X$ and $d(x, x_n) < \varepsilon_{n+1}$, then $d(T^{\lambda_n}(x), T^{\lambda_n}(x_n)) < \varepsilon_n - d(T^{\lambda_n}(x_n), x_{n-1})$. So, whenever $d(x, x_n) < \varepsilon_{n+1}$ we have that

$$d(T^{\lambda_n}(x), x_{n-1}) \leqslant d(T^{\lambda_n}(x), T^{\lambda_n}(x_n)) + d(T^{\lambda_n}(x_n), x_{n-1}) < \varepsilon_n.$$

Since F is B-almost recurrent there exists $\lambda_{n+1} \in B$ with $\lambda_n \prec \lambda_{n+1}$ and $x_{n+1} \in F$ such that $d(T^{\lambda_{n+1}}(x_{n+1}), x_n) < \varepsilon_{n+1}$. This finishes the construction.
We will prove that if $i, j \in \mathbb{N}$ and $i < j$, then

$$d(T^{\lambda_{i+1} * \dots * \lambda_j}(x_j), x_i) < \varepsilon_{i+1}.$$

Indeed, since $d(T^{\lambda_j}(x_j), x_{j-1}) < \varepsilon_j$ we have that $d(T^{\lambda_{j-1}}(T^{\lambda_j}(x_j)), x_{j-2}) < \varepsilon_{j-1}$, and, since $\lambda_{j-1} \prec \lambda_j$, we have that $d(T^{\lambda_{j-1} * \lambda_j}(x_j), x_{j-2}) < \varepsilon_{j-1}$. Repeating the same procedure we obtain that $d(T^{\lambda_{i+1} * \dots * \lambda_j}(x_j), x_i) < \varepsilon_{i+1} \leqslant \varepsilon_1 = \varepsilon/2$.
Since X is compact there exist $i < j$ such that $d(x_i, x_j) < \varepsilon/2$. Then,

$$d(T^{\lambda_{i+1} * \dots * \lambda_j}(x_j), x_j) \leqslant d(T^{\lambda_{i+1} * \dots * \lambda_j}(x_j), x_i) + d(x_i, x_j) < \varepsilon.$$

For $x = x_j$ and $\lambda = \lambda_{i+1} * \dots * \lambda_j \in B$ we have that $\lambda_0 \prec \lambda$ and $d(T^\lambda(x), x) < \varepsilon$. □

We will define now the notion of the recurrent subsets of a compact metric space with respect to a dynamical system indexed by a directed partial semigroup in analogy to the definition of the recurrent elements of the space, in order to locate recurrent elements in them.

Definition 3.11. Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X,d) and $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$.

A closed subset F of X is called B-**recurrent** for $B \in \mathcal{B}$ if for every $\varepsilon > 0$ and every $x \in F$, there exist $A \in \mathcal{B}$ with $A \subseteq B$ and $y \in F$ such that

$$d(T^\lambda(y), x) < \varepsilon, \quad \text{for every} \quad \lambda \in A.$$

A closed subset F of X is called **recurrent** if it is B- recurrent for every $B \in \mathcal{B}$.

Obviously, a B-recurrent subset of X, for $B \in \mathcal{B}$, is B- almost recurrent and, according to Proposition 3.10, it contains B- almost recurrent points. As we will prove in Proposition 3.18 below, we can locate B- recurrent points in homogenous B-recurrent subset of

X. So, toward Proposition 3.18, we will give the appropriate definitions starting from the definition of a minimal dynamical system.

Definition 3.12. Let X be a compact metric space, $(\Lambda, \prec, *)$ a directed partial semigroup and $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of (X, d). This system is **minimal** if no proper closed subset $Y \subset X$ is T^λ-invariant for every $\lambda \in \Lambda$.

Using Zorn's lemma, can be proved that there exists a closed non-empty subset Y of X such that the system $\{T^\lambda\}_{\lambda\in\Lambda}$ restricted to Y to be minimal. According to [11], Lemma 1.14, there exists the following characterization of minimality in case Λ is a semigroup.

Proposition 3.13 ([11])**.** *Let X be a compact metric space, G a semigroup and let $\{T^g\}_{g\in G}$ be a G-dynamical system of X.The dynamical system $\{T^g\}_{g\in G}$ is minimal if for every open subset U of X, there exist finitely many elements $g_1, g_2, \dots, g_n \in G$ such that*

$$\bigcup_{i=1}^{n} (T^{g_i})^{-1}(U) = X.$$

We will give now the definition of a homogenous subset of X with respect to a set $\{T_i\}_{i\in I}$ of transformations acting on X, which introduced by Furstenberg in [11] as follows:

Definition 3.14 ([11])**.** Let X be a compact metric space and F a closed subset of X. Then F is called **homogeneous** with respect to a set of transformations $\{T_i\}_{i\in I}$ acting on X if there exists a group of homeomorphisms G of X each of which commutes with each T_i and such that G leaves F invariant and (F, G) is minimal (no proper closed subset of F is invariant under the action of G).

In the following proposition we will prove that a homogeneous subset of X is recurrent if satisfies a weaker condition than that in Definition 3.11.

Proposition 3.15. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X, d) and $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$. Let $B \in \mathcal{B}$. If a closed subset F of X is homogeneous with respect to the system $\{T^\lambda\}_{\lambda\in\Lambda}$ and for every $\varepsilon > 0$ there exist $x, y \in F$ and $A \in \mathcal{B}$, $A \subseteq B$ such that*

$$d(T^\lambda(y), x) < \varepsilon, \ \ \textit{for every} \ \ \lambda \in A$$

then F is B-recurrent.

Proof. Since F is a homogeneous set with respect to the system $\{T^\lambda\}_{\lambda\in\Lambda}$, there exist a group G of homeomorphisms each of which commutes with each T^λ and such that G leaves F invariant and (F, G) is minimal.
We claim that for every $\varepsilon > 0$ there exists a finite subset G_0 of G such that, for every $x, y \in F$, $\min_{g\in G_0} d(g(x), y) < \varepsilon/2$.

Indeed, let $\{U_i\}_{i=1}^k$ be a finite covering of F by open sets of diameter $< \varepsilon/2$. According to Proposition 3.13, we can find a finite set $\{g_1^i, \dots, g_{m_i}^i\}$ for every $1 \leqslant i \leqslant k$ such that $\bigcup_{j=1}^{m_i} (g_j^i)^{-1}(U_i) = F$. Let $G_0 = \{g_j^i : 1 \leqslant i \leqslant k, \ 1 \leqslant j \leqslant m_i\}$. Then for every $x, y \in F$ we have that $y \in U_{i_0}$ for some $i_0 \in \{1, \dots, \kappa\}$ and $x \in (g_{j_0}^{i_0})^{-1}(U_{i_0})$ for some $j_0 \in \{1, \dots, m_{i_0}\}$. Then $g_{j_0}^{i_0}(x) \in U_{i_0}$ and since U_{i_0} has diameter $< \varepsilon/2$, we have that $\min_{g\in G_0} d(g(x), y) \leqslant d(g_{j_0}^{i_0}(x), y) < \varepsilon/2$. This proves the claim.

Let $\varepsilon > 0$ and $z \in F$. There exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$, then $d(g(x_1), g(x_2)) < \varepsilon/2$ for every $g \in G_0$. According to our hypothesis there exist $x, y \in F$

and $A \in \mathcal{B}$, $A \subseteq B$ such that $d(T^\lambda(y), x) < \delta$ for every $\lambda \in A$. Then $d(g(T^\lambda(y)), g(x)) < \varepsilon/2$ for every $g \in G_0$ and $\lambda \in A$. Since each $g \in G_0$ commutes with each T^λ, we have that for every $g \in G_0$ and $\lambda \in A$, $d(T^\lambda(g(y)), g(x)) = d(g(T^\lambda(y)), g(x)) < \varepsilon/2$.
According to our claim, there exists $g \in G_0$ such that $d(g(x), z) < \varepsilon/2$. Then $d(T^\lambda(g(y)), z) \leqslant d(T^\lambda(g(y)), g(x)) + d(g(x), z) < \varepsilon$, for every $\lambda \in A$. Hence F is B-recurrent, since $A \in \mathcal{B}$, $A \subseteq B$ and $g(y) \in F$, since F is homogeneous. □

In the sequel of the previous proposition we have the following:

Proposition 3.16. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X, d) and $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$. Let $B \in \mathcal{B}$. If a subset F of X is homogeneous with respect to the system $\{T^\lambda\}_{\lambda\in\Lambda}$ and B-recurrent, then for every $\varepsilon > 0$ there exists an element $x_0 \in F$ and $A \in \mathcal{B}$, $A \subseteq B$ such that*

$$d(T^\lambda(x_0), x_0) < \varepsilon, \ \ \textit{for every } \lambda \in A.$$

Proof. Since F is a homogeneous set with respect to the system $\{T^\lambda\}_{\lambda\in\Lambda}$, there exist a group G of homeomorphisms each of which commutes with each T^λ and such that G leaves F invariant and (F, G) is minimal. Using the homogeneity as in the previous proposition, we have that for every $\varepsilon > 0$ there exists a finite subset G_0 of G such that for every $x_1, x_2 \in F$, $\min_{g\in G_0} d(g(x_1), x_2) < \varepsilon/2$.

Let $\varepsilon > 0$. There exists $\delta > 0$ such that if $x_1, x_2 \in X$ and $d(x_1, x_2) < \delta$, then $d(g(x_1), g(x_2)) < \varepsilon/2$ for every $g \in G_0$. Let $x \in F$. Since F is B-recurrent, there exist $A \in \mathcal{B}$ with $A \subseteq B$ and $y \in F$ such that $d(T^\lambda(y), x) < \delta$, for every $\lambda \in A$.

Then, since each $g \in G_0$ commutes with each T^λ, we have that

$$d(T^\lambda(g(y)), g(x)) = d(g(T^\lambda(y)), g(x)) < \varepsilon/2$$

for every $g \in G_0$ and $\lambda \in A$.
Let $g \in G_0$ such that $d(g(x), g(y)) < \varepsilon/2$. Then for every $\lambda \in A$ we have

$$d(T^\lambda(g(y)), g(y)) \leqslant d(T^\lambda(g(y)), g(x)) + d(g(x), g(y)) < \varepsilon.$$

Set $x_0 = g(y) \in F$. □

We will prove, in Proposition 3.18 below, in case the coideal basis $\mathcal{B}$ has the (D)-property and the set $\{T^\lambda : \lambda \in \Lambda\}$ is equicontinuous, the set of B-recurrent elements of a homogeneous recurrent subset F of X is a dense subset of F for every $B \in \mathcal{B}$.

Definition 3.17. We say that a set $\{T_i\}_{i\in I}$ of continuous functions from a compact metric space (X, d) to itself is equicontinuous, if for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $x, y \in X$ with $d(x, y) < \delta$, then $d(T^i(x), T^i(y)) < \varepsilon$ for every $i \in I$.

Proposition 3.18. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\{T^\lambda\}_{\lambda\in\Lambda}$ a Λ-dynamical system of a compact metric space (X, d) which is equicontinuous and $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$ with the (D)-property. Let $B \in \mathcal{B}$. Then every recurrent homogeneous subset F of X contains B-recurrent points. Moreover the set of B-recurrent elements of F is a dense subset of F.*

Proof. Let V be an open subset of X such that $V \cap F \neq \emptyset$. There exists an open set V' such that $V' \subseteq V$, $V' \cap F \neq \emptyset$ and $\delta > 0$ such that if $x \in X$ and $d(x, V') < \delta$, then $x \in V$.

Since F is homogeneous with respect to the system $\{T^\lambda\}_{\lambda\in\Lambda}$, there exist a group G of homeomorphisms commuting with $\{T^\lambda\}_{\lambda\in\Lambda}$ and such that G leaves F invariant and (F, G) is minimal. According to the Proposition 3.13, there exists a finite subset G_0 of G such

that $F \subseteq \bigcup_{g\in G_0} g^{-1}(V')$.
Let $\varepsilon > 0$ such that if $x_1, x_2 \in X$ with $d(x_1, x_2) < \varepsilon$, then $d(g(x_1), g(x_2)) < \delta$ for every $g \in G_0$. Since F is a recurrent and homogeneous subset of X, according to Proposition 3.16, there exists an element $x_0 \in F$ and $A \in \mathcal{B},\ A \subseteq B$ such that

$$d(T^\lambda(x_0), x_0) < \varepsilon, \ \text{ for every } \ \lambda \in A.$$

Let $g \in G_0$ such that $g(x_0) \in V'$. Then $d(T^\lambda(g(x_0)), g(x_0)) < \delta$ for every $\lambda \in A$. Since $g(x_0) \in V'$, we have that $T^\lambda(g(x_0)) \in V$ for every $\lambda \in A$. Hence, for each open set V with $V \cap F \neq \emptyset$ there exist $A \in \mathcal{B},\ A \subseteq B$ and $x' = g(x_0) \in V \cap F$ such that $T^\lambda(x') \in V$ for every $\lambda \in A$.

Consequently, since $\{T^\lambda\}_{\lambda\in\Lambda}$ is equicontinuous, it follows that for every open set V with $V \cap F \neq \emptyset$ there exist $A \in \mathcal{B},\ A \subseteq B$ and an open set V_1 such that

$$V_1 \cap F \neq \emptyset, \overline{V_1} \subseteq V \text{ and } T^\lambda(V_1) \subseteq V \text{ for every } \lambda \in A.$$

Let V_0 be an open subset of X such that $V_0 \cap F \neq \emptyset$. Inductively we construct a sequence $(V_n)_{n\in\mathbb{N}}$ of open sets and also a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathcal{B}$, with $B \supseteq A_1 \supseteq A_2 \supseteq \ldots$ such that for every $n \in \mathbb{N}$

$$\overline{V_n} \subseteq V_{n-1}, \ \ V_n \cap F \neq \emptyset \ \ \textit{and} \ \ T^\lambda(V_n) \subseteq V_{n-1}, \ \ \textit{for every} \ \ \lambda \in A_n.$$

We can also suppose that the diameter of V_n tends to 0. Let $\bigcap_{n\in\mathbb{N}} \overline{V_n} \cap F = \{x_0\}$. Then $x_0 \in V_0$ and we will prove that x_0 is a B-recurrent element of F.

Indeed, since the coideal basis $\mathcal{B}$ has the (D)-property, there exists $C \in \mathcal{B}$, such that for every $n \in \mathbb{N}$ there exists $k_n \in \mathbb{N} \cup \{0\}$ such that

$$k_n = \max\{k \in \mathbb{N} : \text{there exist } \lambda_1, \ldots, \lambda_k \in C \setminus A_n \text{ with } \lambda_1 \prec \cdots \prec \lambda_k\}.$$

This implies that $C \setminus A_1$ does not contain an element of $\mathcal{B}$. So, since $C \in \mathcal{B}$, there exists $A \in \mathcal{B},\ A \subseteq C \cap A_1$. Then $A \subseteq C,\ A \in \mathcal{B},\ A \subseteq A_1 \subseteq B$ and for every $n \in \mathbb{N},\ n > 1$ there exists $q_n \in \mathbb{N} \cup \{0\}$ such that

$$q_n = \max\{k \in \mathbb{N} : \text{there exist } \lambda_1, \ldots, \lambda_k \in A \setminus A_n \text{ with } \lambda_1 \prec \cdots \prec \lambda_k\}.$$

We will prove that $\lim_{\lambda\in A} T^\lambda(x_0) = x_0$. Let $\varepsilon > 0$. Since the diameter of V_n tends to 0, pick $n_0 \in \mathbb{N}, n_0 > 1$ such that the diameter of V_{n_0} to be less than ε. Let $\lambda_1, \ldots, \lambda_{q_{n_0+1}} \in A \setminus A_{n_0+1}$ with $\lambda_1 \prec \cdots \prec \lambda_{q_{n_0+1}}$. Let $\lambda_0 \in A \cap A_{n_0+1}$ such that $\lambda_{q_{n_0+1}} \prec \lambda_0$. For every $\lambda \in A$ with $\lambda_0 \prec \lambda$ we have that $\lambda \in A \cap A_{n_0+1}$ and consequently that $T^\lambda(x_0) \in V_{n_0}$. Since $x_0 \in V_{n_0}$, we have that $d(T^\lambda(x_0), x_0) < \varepsilon$ for every $\lambda \in A$ with $\lambda_0 \prec \lambda$. Hence, x_0 is a B-recurrent elements of F and $x_0 \in V_0$. This finishes the proof. □

Finally, using the previous proposition, we will prove,under some additional hypotheses, a multiple recurrence Theorem analogous of the starting Theorem 3.6.

Theorem 3.19. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$ with the (D)-property and $\{T_1^\lambda\}_{\lambda\in\Lambda}, \ldots, \{T_m^\lambda\}_{\lambda\in\Lambda}$ be Λ-dynamical systems of a compact metric space (X, d), all contained in a commutative group G of homeomorphisms of X, and let the systems $\{T_i^\lambda\}_{\lambda\in\Lambda}$, $\{(T_i^\lambda)^{-1}\}_{\lambda\in\Lambda}$ be equicontinuous for each $i = 1, \ldots, m$. Then, for every $B \in \mathcal{B}$ there exist $A \in \mathcal{B}$ with $A \subseteq B$ and $x_0 \in X$ such that*

$$\lim_{\lambda\in A} T_i^\lambda(x_0) = x_0 \ \ \textit{for every} \ \ 1 \leqslant i \leqslant m.$$

Proof. We can assume without loss of generality that (X, G) is minimal, otherwise we can replace X by a G-minimal subset of X. We proceed by induction on m. For $m = 1$ the Theorem is valid from Theorem 3.6. Assume that the theorem holds for some $m \in \mathbb{N}$. Let $B \in \mathcal{B}$ and $\{T_1^\lambda\}_{\lambda\in\Lambda}, \ldots, \{T_{m+1}^\lambda\}_{\lambda\in\Lambda}$ be $m+1$ such systems. We set

$$S_i^\lambda = T_i^\lambda \circ (T_{m+1}^\lambda)^{-1}, \text{ for all } 1 \leqslant i \leqslant m.$$

We note that, since G is a commutative group, $S_i^{\lambda_1 * \lambda_2} = S_i^{\lambda_1} \circ S_i^{\lambda_2}$ for every $\lambda_1, \lambda_2 \in B$ with $\lambda_1 \prec \lambda_2$ and $1 \leqslant i \leqslant m$. Hence, $\{S_1^\lambda\}_{\lambda\in\Lambda}, \ldots, \{S_m^\lambda\}_{\lambda\in\Lambda}$ are Λ-dynamical systems of (X, d) satisfying the hypotheses of the theorem. Applying the induction hypothesis , we have the existence of $y_0 \in X$ and $A \in \mathcal{B}$ with $A \subseteq B$ such that $\lim_{\lambda\in A} S_i^\lambda(y_0) = y_0$, for every $1 \leqslant i \leqslant m$.

Let $\varepsilon > 0$. For each $i = 1, \ldots, m$ there exists $\lambda_i \in A$ such that

$$d(T_i^\lambda((T_{m+1}^\lambda)^{-1}(y_0)), y_0) = d(S_i^\lambda(y_0), y_0) < \varepsilon/2, \text{ for every } \lambda \in A \text{ with } \lambda_i \prec \lambda.$$

Let $\lambda_0 \in A$ with $\lambda_1, \ldots, \lambda_m \prec \lambda_0$. Then for every $\lambda \in A$ with $\lambda_0 \prec \lambda$ we have that

$$d(T_i^\lambda((T_{m+1}^\lambda)^{-1}(y_0)), y_0) < \varepsilon/2, \text{ for every } i = 1, \ldots, m+1.$$

According to Theorem 2.10, there exists $y_1 \in X$ and $A_1 \in \mathcal{B}$, $A_1 \subseteq A$ such that $\lim_{\lambda\in A_1} (T_{m+1}^\lambda)^{-1}(y_0) = y_1$.

Since $\{T_i^\lambda\}_{\lambda\in\Lambda}$, for $i = 1, \ldots, m+1$, are equicontinuous systems, there exists $\delta > 0$ such that if $x, y \in X$ with $d(x, y) < \delta$, then $d(T_i^\lambda(x), T_i^\lambda(y)) < \varepsilon/2$ for every $\lambda \in \Lambda$ and $i = 1, \ldots, m+1$. Let $\lambda_1 \in A_1$ with $\lambda_0 \prec \lambda_1$ such that $d((T_{m+1}^\lambda)^{-1}(y_0), y_1) < \delta$, for every $\lambda \in A_1$ with $\lambda_1 \prec \lambda$. Hence, $d(T_i^\lambda((T_{m+1}^\lambda)^{-1}(y_0)), T_i^\lambda(y_1)) < \varepsilon/2$, for every $\lambda \in A_1$ with $\lambda_1 \prec \lambda$ and every $i = 1, \ldots, m+1$. Then for every $\lambda \in A_1$ with $\lambda_1 \prec \lambda$ and every $i = 1, \ldots, m+1$ we have

$$d(T_i^\lambda(y_1), y_0) \leqslant d(T_i^\lambda(y_1), T_i^\lambda((T_{m+1}^\lambda)^{-1}(y_0))) + d(T_i^\lambda((T_{m+1}^\lambda)^{-1}(y_0)), y_0) < \varepsilon.$$

Let $C \in \mathcal{B}$, $C \subseteq A_1 - \lambda_1 \subseteq B$. Then

$$\max\{d(T_i^\lambda(y_1), y_0) \;:\; i = 1, \ldots, m+1\} < \varepsilon, \text{ for every } \lambda \in C.$$

Let the compact metric space $(X^{m+1}, \widetilde{d})$, where

$$\widetilde{d}((y_1, \ldots, y_{m+1}), (x_1, \ldots, x_{m+1})) = \max\{d((y_i, x_i)), i = 1, \ldots, m+1\},$$

and the Λ-dynamical system $\{\widetilde{T}^\lambda\}_{\lambda\in\Lambda}$ of X^{m+1}, where $\widetilde{T}^\lambda = T_1^\lambda \times \cdots \times T_{m+1}^\lambda$, which is equicontinuous. Let $\Delta^{m+1} = \{(x, \ldots, x) : x \in X\} \subseteq X^{m+1}$ be the diagonal subset of X^{m+1}. We can assume that G acts on X^{m+1} by replacing each $g \in G$ with $g \times \ldots \times g$. Then the functions $\widetilde{T}^\lambda$, for $\lambda \in \Lambda$, commute with the functions of G, G leaves Δ^{m+1} invariant and (Δ^{m+1}, G) is minimal. Hence, Δ^{m+1} is a homogeneous set with respect to $(\widetilde{T}^\lambda)_{\lambda\in\Lambda}$.

According to Proposition 3.18, in order to prove the theorem, it will suffice to prove that Δ^{m+1} is B-recurrent. Hence, according to Proposition 3.15, it is enough for a given $\varepsilon > 0$ to find $x, y \in X$ and $C \in \mathcal{B}, C \subseteq B$ such that

$$\widetilde{d}(\widetilde{T}^\lambda((y, \ldots, y)), (x, \ldots, x)) = \max\{d(T_i^\lambda(y), x) \;:\; i = 1, \ldots, m+1\} < \varepsilon, \text{ for every } \lambda \in C.$$

But we have already proved that, for a given $\varepsilon > 0$ there exists $y_1, y_0 \in X$ and $C \in \mathcal{B}, C \subseteq B$ such that

$$\max\{d(T_i^\lambda(y_1), y_0) \;:\; i = 1, \ldots, m+1\} < \varepsilon, \text{ for every } \lambda \in C.$$

Hence Δ^{m+1} is a recurrent set. This finishes the proof. $\square$

Corollary 3.20. *Let $(\Lambda, \prec, *)$ be a directed partial semigroup, $\mathcal{B}$ a suitable coideal basis on $(\Lambda, \prec, *)$ with the (D)-property and $\{T_1^\lambda\}_{\lambda\in\Lambda}, \ldots, \{T_m^\lambda\}_{\lambda\in\Lambda}$ Λ-dynamical systems of a compact metric space (X, d) all contained in a commutative group G of homeomorphisms of X and let $\{T_i^\lambda\}_{\lambda\in\Lambda}$, $\{(T_i^\lambda)^{-1}\}_{\lambda\in\Lambda}$ be equicontinuous for each $i = 1, \ldots, m$. For every non-empty open subset U of X, there exist $x_0 \in X$ and $C \in \mathcal{B}$ with $C \subseteq B$ such that*

$$\bigcap_{i=1}^{m} (T_i^\lambda)^{-1}(U) \neq \emptyset, \text{ for every } \lambda \in C.$$

Proof. Since G acts minimally on X, according to Proposition 3.13, there exists a finite subset G_0 of G such that $X = \bigcup_{g\in G_0} g^{-1}(U)$. According to Theorem 3.19, there exist $x_0 \in X$ and $A \in \mathcal{B}$ with $A \subseteq B$ such that

$$\lim_{\lambda\in A} T_i^\lambda(x_0) = x_0, \text{ for every } 1 \leqslant i \leqslant m.$$

Let $g \in G_0$ such that $x_0 \in g^{-1}(U)$. Then, there exists $\lambda_0 \in A$ such that $T_i^\lambda(x_0) \in g^{-1}(U)$, for every $\lambda \in A$ with $\lambda_0 \prec \lambda$ and for every $1 \leqslant i \leqslant m$. Let $C \in \mathcal{B}$, $C \subseteq A - \lambda_0 \subseteq B$. Hence, $g(x_0) \in \bigcap_{i=1}^{m} (T_i^\lambda)^{-1}(U)$, for every $\lambda \in C$.

$\square$

4. Applications to semigroups with digital representation

We will indicate a way to apply the recurrence results for topological dynamical systems or nets proved in the previous sections to systems or nets indexed by semigroups with digital representation. So, we will define a relation on a semigroup with digital representation in order to make it a directed partial semigroup. In order to define a suitable coideal basis satisfying the (D)-property on a semigroup with digital representation $< D_i >_{i\in I}$ we will introduce the $< D_i >_{i\in I}$-located words. So, the recurrent results for topological dynamical systems or nets proved in the previous sections, can be applied to systems or nets indexed by $< D_i >_{i\in I}$-located words or semigroups with digital representation. Moreover, Proposition 3.18 and the multiple recurrence Theorem 3.19 for these systems hold without the assumption of the equicontinuity of the systems.

The notion of a semigroup with digital representation introduced by Ferri, Hindman and Strauss in [10] as follows:

Definition 4.1. ([10]) A semigroup $(X, +)$ has a **digital representation** $< D_i >_{i\in I}$, where I is a linearly ordered set, D_i is a non-empty finite subset of X for every $i \in I$ and each element of X is uniquely representable as a sum $\sum_{i\in H} x_i$, where H is a finite subset of I, $x_i \in D_i$ for every $i \in H$ and sums are taken in increasing order of indices.

If X has an identity 0_X, then we set $0_X = \sum_{i\in\emptyset} x_i$.

In order to make an infinite semigroup $(X, +)$ with a digital representation $< D_i >_{i\in I}$ a directed partial semigroup, we will endow the set $[I]^{<\infty}$ of all the finite subsets of I with an appropriate relation.

Definition 4.2. Let I be an infinite linearly ordered set. A relation $<_R$ on the set $[I]^{<\infty}$ of all the finite subsets of I is called a **proper relation** on $[I]^{<\infty}$ if satisfies:

(i) $\emptyset <_R H$, $H <_R \emptyset$ for every $H \in [I]^{<\infty}_{>0}$.

(ii) If $H_1, H_2 \in [I]^{<\infty}_{>0}$ and $H_1 <_R H_2$, then, for each $i \in H_2$, either $i > maxH_1$ or $i < \min H_1$.

(ii) $([I]^{<\infty}, <_R, \cup)$ is a directed partial semigroup.

Let a semigroup $(X, +)$ with a digital representation $< D_i >_{i\in I}$, where I is an infinite linearly ordered set, and let $<_R$ a proper relation on $[I]^{<\infty}$. We define for $s_1 = \sum_{i\in H_1} x_i, s_2 = \sum_{i\in H_2} x_i \in X$, where $H_1, H_2 \in [I]^{<\infty}$,

$$s_1 <_R s_2 \Longleftrightarrow H_1 <_R H_2.$$

Moreover, for $s_1 = \sum_{i\in H_1} x_i, s_2 = \sum_{i\in H_2} x_i \in X$ with $s_1 <_R s_2$ we define the concatination

$$s_1 \star s_2 = \textstyle\sum_{i\in H_1\cup H_2} x_i.$$

The following proposition holds:

Proposition 4.3. *Let a semigroup $(X, +)$ with a digital representation $< D_i >_{i\in I}$, where I is an infinite linearly ordered set, and let $<_R$ a proper relation on $[I]^{<\infty}$. Then $(X, <_R, \star)$ is a directed partial semigroup.*

We will give some examples of semigroups with digital representation.

Examples 4.4. **(1)** Let $p \in \mathbb{N}$, $p > 1$. The semigroup $(\mathbb{N}, +)$ has a digital representation $< D_n >_{n\in\mathbb{N}}$, where $D_n = \{ip^{n-1} : 1 \le i \le p-1\}$. For $H_1, H_2 \in [\mathbb{N}]^{<\infty}_{>0}$, we define $H_1 <_R H_2$ if and only if $\max H_1 < \min H_2$. Then $(\mathbb{N}, <_R, \star)$ is a directed partial semigroup.

(2) Let a sequence $(k_n)_{n\in\mathbb{N}}$ of natural numbers. According to [3], the semigroup $(\mathbb{Z}, +)$ has a digital representation $< D_n >_{n\in\mathbb{N}}$, where, $D_1 = \{1\}$ and, for each $n \in \mathbb{N}, n \ge 2$, $D_n = \{i(-1)^{n+1}(k_1+1)\cdots(k_{n-1}+1) : 1 \le i \le k_n\}$. If $<_R$ is the relation on $[\mathbb{N}]^{<\infty}$ defined in the previous example, then $(\mathbb{Z}, <_R, \star)$ is a directed partial semigroup.

(3) More general, if a semigroup $(X, +)$ has a digital representation $< D_i >_{i\in I}$, where I is an infinite linearly ordered set and for every $i \in I$ there exists $j \in I$ with $i < j$, then, defining $H_1 <_R H_2$, for $H_1, H_2 \in [I]^{<\infty}_{>0}$, if and only if $\max H_1 < \min H_2$, we can make $(X, <_R, \star)$ a directed partial semigroup.

(4) The semigroup $(\mathbb{Q}, +)$, according to [3], has a digital representation $< D_n >_{n\in\mathbb{Z}}$, where $D_n = \{i(-1)^n(n+1)! : 1 \le i \le n+1\}$ for $n \in \mathbb{N}\cup\{0\}$ and for $n \in \mathbb{Z}^-$, $D_n = \{i\frac{(-1)^{-n}}{(-n+1)!} : 1 \le i \le -n\}$. For $H_1, H_2 \in [\mathbb{Z}]^{<\infty}_{>0}$, we define $H_1 <_R H_2$ if and only if $H_2 = A_1 \cup A_2$ with $A_1, A_2 \neq \emptyset$ and $\max A_1 < \min H_1$, $\max H_1 < \min A_2$. Then $(\mathbb{Q}, <_R, \star)$, is a directed partial semigroup.

In order to define a suitable coideal basis satisfying the (D)-property on a semigroup $(X, +)$ with digital representation $< D_i >_{i\in I}$ we will introduce the $< D_i >_{i\in I}$-located words.

Definition 4.5. Let an arbitrary alphabet Σ, an infinite linearly ordered set $(I, <)$ and, for each $i \in I$, let $D_i = \{d_{1,i}, \ldots, d_{k_i,i}\}$ be a non-empty finite subset of Σ with cardinality $k_i \in \mathbb{N}$. We define the set of (constant) $(D_i)_{i\in I}$**-located words** as follows:

$$L((D_i)_{i\in I}) = \{w = w_{i_1}\ldots w_{i_l} : l \in \mathbb{N}, i_1 < \ldots < i_l \in I \text{ and } w_{i_j} \in D_{i_j} \ \forall\, 1 \le j \le l\}.$$

Let $m \in \mathbb{N}$ and $\vec{\upsilon} = (\upsilon_1, \ldots, \upsilon_m)$, where $\upsilon_1, \ldots, \upsilon_m \notin \Sigma$ be the variables. We define the set of **variable** $(D_i)_{i\in I}$**-located words** as follows:

$$L((D_i)_{i\in I}; \upsilon_1, \ldots, \upsilon_m) = \{w = w_{i_1}\ldots w_{i_l} : l \in \mathbb{N}, i_1 < \ldots < i_l \in I, w_{i_j} \in D_{i_j} \cup \{\upsilon_1, \ldots, \upsilon_m\} \ \forall\, 1 \le j \le l \text{ and there exist } 1 \le j_1, \ldots, j_m \le l \text{ with } w_{i_{j_1}} = \upsilon_1, \ldots, w_{i_{j_m}} = \upsilon_m\}.$$

Let $L = L((D_i)_{i\in I}) \cup L((D_i)_{i\in I}; \vec{\upsilon})$. For $w = w_{i_1}\ldots w_{i_l} \in L$, the set $dom(w) = \{i_1 < \ldots < i_l\} \subseteq I$ is the *domain* of w.

We assume that there exists a proper relation $<_R$ on the set $[I]^{<\infty}$ of all the finite, subsets of I. Then we define for $w, u \in L$ the relation

$$w <_R u \Longleftrightarrow dom(w) <_R dom(u).$$

and also for two words $w = w_{i_1} \dots w_{i_l}, u = u_{t_1} \dots u_{t_k} \in L$ with $w <_R u$ we define the concatenating word

$$w \star u = z_{q_1} \dots z_{q_{r+l}} \in L,$$

where $\{q_1 < \dots < q_{r+l}\} = dom(w) \cup dom(u)$, $z_i = w_i$ if $i \in dom(w)$ and $z_i = u_i$ if $i \in dom(u)$.

So, the following proposition holds:

Proposition 4.6. *Let an arbitrary alphabet Σ, an infinite linearly ordered set $(I, <)$ and let $< D_i >_{i \in I}$ be a family of non-empty finite subsets of Σ. If $<_R$ is a proper relation on the set $[I]^{<\infty}$ and $m \in \mathbb{N}$, then $(L((D_i)_{i\in I}), <_R, \star)$ and $(L((D_i)_{i\in I}; \upsilon_1, \dots, \upsilon_m), <_R, \star)$ are directed partial semigroups.*

Let $m \in \mathbb{N}$ and $w = w_{i_1} \dots w_{i_l} \in L((D_i)_{i\in I}; \upsilon_1, \dots, \upsilon_m)$, where $D_i = \{d_{1,i}, \dots, d_{k_i,i}\}$. For $(p_1, \dots, p_m) \in \mathbb{N}^m \cup \{(0, \dots, 0)\}$ we set $w(0, \dots, 0) = w$ and for $(p_1, \dots, p_m) \in \mathbb{N}^m$

$$w(p_1, \dots, p_m) = u_{i_1} \dots u_{i_l} \in L((D_i)_{i\in I}),$$

where, for $1 \le j \le l$, $u_{i_j} = w_{i_j}$ if $w_{i_j} \in D_{i_j}$, $u_{i_j} = d_{p_r, i_j}$ if $w_{i_j} = \upsilon_r$, for $1 \le r \le m$, and $p_r \le k_{i_j}$ and finally $u_{i_j} = d_{k_{i_j}, i_j}$ if $w_{i_j} = \upsilon_r$, for $1 \le r \le m$, and $p_r > k_{i_j}$. We set

$$L^\omega((D_i)_{i\in I}; \upsilon_1, \dots, \upsilon_m) = \{\vec{w} = (w_n)_{n\in\mathbb{N}} : \ w_n \in L((D_i)_{i\in I}; \upsilon_1, \dots, \upsilon_m) \text{ and } w_n <_R w_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

Extracted Elements, Extractions. Let $m \in \mathbb{N}$. We fix an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}^m$ such that $\bigcup_{n\in\mathbb{N}} F_n = \mathbb{N}^m$. Let a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega((D_i)_{i\in I}; \upsilon_1, \dots, \upsilon_m)$.

An **extracted variable $(D_i)_{i\in I}$-located word** of $\vec{w}$ has the form

$$u = w_{n_1}(\vec{p_1}) \star \dots \star w_{n_\lambda}(\vec{p_\lambda}) \in L((D_i)_{i\in I}; \upsilon_1, \dots, \upsilon_m),$$

where $\lambda \in \mathbb{N}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$, $\vec{p_i} \in F_{n_i} \cup \{(0, \dots, 0)\}$ for every $1 \le i \le \lambda$ and $(0, \dots, 0) \in \{\vec{p_1}, \dots, \vec{p_\lambda}\}$. The set of all the extracted variable $(D_i)_{i\in I}$-located words of $\vec{w}$ is denoted by $EV(\vec{w})$.

An **extracted $(D_i)_{i\in I}$-located word** of $\vec{w}$ has the form

$$z = w_{n_1}(\vec{p_1}) \star \dots \star w_{n_\lambda}(\vec{p_\lambda}) \in L((D_i)_{i\in I}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$ and $\vec{p_i} \in F_{n_i}$ for every $1 \le i \le \lambda$. The set of all the extracted $(D_i)_{i\in I}$-located words of $\vec{w}$ is denoted by $E^m(\vec{w})$.

Remark 4.7. *A special case of $< D_i >_{i\in I}$-located words is the ω-located words, defined in Example 2.9, (3), where $\Sigma = \{\alpha_n : \ n \in \mathbb{N}\}$, $\upsilon \notin \Sigma$, $I = \mathbb{N}$ and $D_n = \{\alpha_1, \dots, \alpha_{k_n}\}$, for every $n \in \mathbb{N}$, where $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ is an increasing sequence. According to a fundamental partition theorem of Carlson, the families*

$$\mathcal{B} = \{E(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)\} \text{ and}$$
$$\mathcal{B}_1 = \{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\omega(\Sigma, \vec{k}; \upsilon)\}$$

are suitable coideal bases on $(L(\Sigma, \vec{k}), \prec)$ and $(L(\Sigma, \vec{k}; \upsilon), \prec)$ respectively. Moreover, as proved in Example 2.9, (3), these coideal bases have the (D)-property.

As a consequence of Carlson's theorem, we will prove an analogous result for $< D_i >_{i\in I}$-located words.

Theorem 4.8. *Let an arbitrary alphabet Σ, an infinite linearly ordered set $(I,<)$, a family $< D_i >_{i\in I}$ of non-empty finite subsets of Σ, a proper relation $<_R$ on the set $[I]^{<\infty}$, $m\in\mathbb{N}$ and an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}^m$ such that $\bigcup_{n\in\mathbb{N}} F_n=\mathbb{N}^m$. The families*

$$\mathcal{B}_1=\{EV(\vec{w}) : \vec{w}\in L^\omega((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m)\} \text{ and}$$
$$\mathcal{B}=\{E(\vec{w}) : \vec{w}\in L^\omega((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m)\}$$

are suitable coideal bases on $(L((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m),<_R,\star)$ and $(L((D_i)_{i\in I}),<_R,\star)$ respectively, and satisfy the (D)-property.

Proof. Let $\vec{w}\in L^\omega((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m)$, and let $EV(\vec{w})=A_1\cup A_2$ and $E(\vec{w})=B_1\cup B_2$. Firstly, we will define an order on the set $\mathbb{N}^m$. For $\vec{p}\in\mathbb{N}^m$ we set $i(\vec{p})$ to be the least $n\in\mathbb{N}$ such that $\vec{p}\in F_n$ and then we define $\vec{p}_1<_*\vec{p}_2$ for $\vec{p}_1,\vec{p}_2\in\mathbb{N}^m$ if and only if either $i(\vec{p}_1)<i(\vec{p}_2)$ or $i(\vec{p}_1)=i(\vec{p}_2)$ and $\vec{p}_1$ is less than $\vec{p}_2$ in the lexicographical ordering.

Let $\mathbb{N}^m=\{\beta_1<_*\beta_2<_*\beta_3<_*\ldots\}$. For each $n\in\mathbb{N}$, let $\beta_{k_n}\in\mathbb{N}^m$, be the greatest element of F_n in the lexicographical ordering. Then $\vec{k}=(k_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}$ is an increasing sequence. We set $\Sigma_1=\{\beta_n:\ n\in\mathbb{N}\}=\mathbb{N}^m$ and we define the function $h:L(\Sigma_1,\vec{k})\cup L(\Sigma_1,\vec{k};\upsilon)\to E(\vec{w})\cup EV(\vec{w})$ with

$$h(t_{n_1}\ldots t_{n_\lambda})=w_{n_1}(p_1^1,\ldots,p_m^1)\star\ldots\star w_{n_\lambda}(p_1^\lambda,\ldots,p_m^\lambda),$$

where, for $1\le i\le\lambda$, $(p_1^i,\ldots,p_m^i)=(0,\ldots,0)$ if $t_{n_i}=\upsilon$ and $(p_1^i,\ldots,p_m^i)=t_{n_i}\in\{\beta_1,\cdots\beta_{k_{n_i}}\}$ if $t_{n_i}\in\Sigma_1$. The function h is onto $E(\vec{w})\cup EV(\vec{w})$, and moreover $h(L(\Sigma_1,\vec{k}))=E(\vec{w})$ and $h(L(\Sigma_1,\vec{k};\upsilon))=EV(\vec{w})$.

According to Carlson's theorem, there exist a sequence $\vec{s}=(s_n)_{n\in\mathbb{N}}\in L^\omega(\Sigma_1,\vec{k};\upsilon)$ and $i_0\in\{1,2\}$, $j_0\in\{1,2\}$ such that $EV(\vec{s})\subseteq h^{-1}(A_{i_0})$ and $E(\vec{s})\subseteq h^{-1}(B_{j_0})$. Set $u_n=h(s_n)\in EV(\vec{w})$ for every $n\in\mathbb{N}$ and $\vec{u}=(u_n)_{n\in\mathbb{N}}\in L^\omega((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m)$. Then $EV(\vec{u})\subseteq h(EV(\vec{s}))\subseteq A_{i_0}$ and $E(\vec{u})\subseteq h(E(\vec{s}))\subseteq B_{j_0}$. Hence, $\mathcal{B}_1$ and $\mathcal{B}$ are coideal bases on $(L((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m),<_R,\star)$ and $(L((D_i)_{i\in I}),<_R,\star)$ respectively, and obviously they are suitable. Analogously to Example 2.9,(3), can be proved that $\mathcal{B}_1$ and $\mathcal{B}$ satisfy the (D)-property. □

We will go back to semigroups with digital representation. Let a semigroup $(X,+)$ has a digital representation $< D_i >_{i\in I}$ and let on the set $[I]^{<\infty}$ is defined a proper relation $<_R$. According to Proposition 4.3, $(X,<_R,\star)$ is a directed partial semigroup. Let $g:L((D_i)_{i\in I})\to X\setminus\{0_X\}$ in case $(X,+)$ has an identity 0_X or $g:L((D_i)_{i\in I})\to X$ otherwise, with

$$g(w_{i_1}\ldots w_{i_l})=w_{i_1}+\ldots+w_{i_l}.$$

The function g is one-to-one, onto, preserves the order and for $w,u\in L((D_i)_{i\in I})$ with $w<_R u$ we have $g(w\star u)=g(w)\star g(u)$. So, via the function g, we can define a suitable coideal basis for $(X,<_R,\star)$ satisfying the (D)- property, using the previous theorem.

Theorem 4.9. *Let a semigroup $(X,+)$ with a digital representation $< D_i >_{i\in I}$ and let a proper relation $<_R$ on the set $[I]^{<\infty}$. Fixing an increasing sequence $(F_n)_{n\in\mathbb{N}}$ of non-empty finite subsets of $\mathbb{N}^m$, for $m\in\mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} F_n=\mathbb{N}^m$, the family*

$$\mathcal{B}=\{g(E(\vec{w})) : \vec{w}\in L^\infty((D_i)_{i\in I};\upsilon_1,\ldots,\upsilon_m)\}$$

is a suitable coideal basis on $(X,<_R,\star)$ satisfying the (D)- property.

So, the recurrent results for topological dynamical systems or nets proved in the previous sections, can be applied to systems or nets indexed by $< D_i >_{i\in I}$-located words or semigroups with digital representation. Moreover, Proposition 3.18 and the multiple recurrence Theorem 3.19 for these dynamical systems hold without the assumption of the equicontinuity of the systems.

Proposition 4.10. *Let a directed partial semigroup* $(L((D_i)_{i\in I}), <_R, \star)$ *and* $\{T^w\}_{w\in L((D_i)_{i\in I})}$ *be a* $L((D_i)_{i\in I})$*-dynamical system of a compact metric space* (X,d)*. We fix* $m\in\mathbb{N}$ *and an increasing sequence* $(F_n)_{n\in\mathbb{N}}$ *of non-empty finite subsets of* $\mathbb{N}^m$ *such that* $\bigcup_{n\in\mathbb{N}} F_n = \mathbb{N}^m$*. Let the coideal basis* $\mathcal{B} = \{E(\vec{u}) : \vec{u}\in L^\omega((D_i)_{i\in I}; \upsilon_1,\dots,\upsilon_m)\}$ *and let* $B = E(\vec{w})\in\mathcal{B}$*. Then every* B*-recurrent homogeneous subset* F *of* X *contains* B*-recurrent points. Moreover the set of* B*-recurrent elements of* F *is a dense subset of* F*.*

Proof. According to the proof of Proposition 3.18, for each open set $V\subseteq X$ with $V\cap F\neq\emptyset$ there exist $A\in\mathcal{B}$, $A\subseteq B$ and $x'\in V\cap F$ such that $T^w(x')\in V$ for every $w\in A$. Since T^w are continuous, for every open set V with $V\cap F\neq\emptyset$ and every $i\in I$, $k\in\mathbb{N}$ there exists an open set V_1 and $u\in EV(\vec{w})$ with $\min dom(u) > i$ such that $\overline{V_1}\subseteq V$ and $T^{u(p_1,\dots,p_m)}(V_1)\subseteq V$ for every $1\le p_1,\dots,p_m\le k$.

Let V_0 be an open subset of X such that $V_0\cap F\neq\emptyset$. Inductively we can define a sequence $(V_n)_{n\in\mathbb{N}}$ of open sets and a sequence $\vec{u}=(u_n)_{n\in\mathbb{N}}\in L^\omega((D_i)_{i\in I};\upsilon_1,\dots,\upsilon_m)$ with $\vec{u}\subseteq EV(\vec{w})$ such that $\overline{V_n}\subseteq V_{n-1}$, $V_n\cap A\neq\emptyset$ and $T^{u_n(p_1^n,\dots,p_m^n)}V_n\subseteq V_{n-1}$ for every $n\in\mathbb{N}$ and $(p_1^n,\dots,p_m^n)\in F_n$. We can also suppose that the diameter of V_n tends to 0. Then $\bigcap_{n\in\mathbb{N}} V_n\cap A = \{x_0\}$.

For $1<s_1<\dots<s_r$, we have that $T^{u_{s_1}(p_1^{s_1},\dots,p_m^{s_1})\star\dots\star u_{s_r}(p_1^{s_r},\dots,p_m^{s_r})}V_{s_r}\subseteq V_{s_1-1}$. Then $T^w(x_0)\in V_i$ for every $w\in E(\vec{u})$ with $u_{i+1} <_R w$. Let $A=E(\vec{u})\subseteq E(\vec{w}) = B$, Then $\lim_{w\in A} T^w(x_0) = x_0$ Hence, $x_0\in A\cap V_0$ is a B-recurrent point. This gives that the set of B-recurrent points in F is dense in F. ☐

Theorem 4.11. *Let a directed partial semigroup* $(L((D_i)_{i\in I}), <_R, \star)$ *and let* r *dynamical systems* $\{T_1^w\}_{w\in L((D_i)_{i\in I})},\dots,\{T_r^w\}_{w\in L((D_i)_{i\in I})}$ *of a compact metric space* (X,d) *all contained in a commutative group* G *of homeomorphisms of* X*. We fix* $m\in\mathbb{N}$ *and an increasing sequence* $(F_n)_{n\in\mathbb{N}}$ *of non-empty finite subsets of* $\mathbb{N}^m$ *such that* $\bigcup_{n\in\mathbb{N}} F_n=\mathbb{N}^m$*. Let the coideal basis* $\mathcal{B}=\{E(\vec{u}) : \vec{u}\in L^\omega((D_i)_{i\in I};\upsilon_1,\dots,\upsilon_m)\}$ *and let* $B=E(\vec{w})\in\mathcal{B}$*. Then, there exist* $x_0\in X$ *and* $\vec{u}\in L^\omega((D_i)_{i\in I};\upsilon_1,\dots,\upsilon_m)$ *with* $\vec{u}\subseteq EV(\vec{w})$ *such that*

$$\lim_{w\in E(\vec{u}} T_i^w(x_0) = x_0 \ \ \textit{for every} \ \ 1\leqslant i\leqslant r.$$

Proof. The proof is analogous to the proof of Theorem 3.19. We assume that (X,G) is minimal and we proceed by induction on $r\in\mathbb{N}$. For $r=1$ we obtain the assertion from Theorem 4.8, analogously to Theorem 3.6. If $\{T_1^w\}_{w\in L((D_i)_{i\in I})},\dots,\{T_{r+1}^w\}_{w\in L((D_i)_{i\in I})}$ are $r+1$ such systems, we set $S_j^w = T_j^w(T_{r+1}^w)^{-1}$ for all $1\le j\le r$. By the induction hypothesis there exist $y\in X$ and $\vec{u}\in L^\omega((D_i)_{i\in I};\upsilon_1,\dots,\upsilon_m)$ with $\vec{u}\subseteq EV(\vec{w})$, and consequently $A=E(\vec{u})\subseteq B=E(\vec{w})$, such that $\lim_{w\in E(\vec{u})} S_j^w(y) = y$ for every $1\le j\le r$.

Consider the diagonal Δ^{r+1} of the product X^{r+1}. We can assume that G acts on X^{r+1}. Also, the functions $\widetilde{T}^w = T_1^w\times\dots\times T_{r+1}^w$ act on X^{r+1} and commute with the functions of G. Hence, Δ^{r+1} is a homogeneous set with respect to $(\widetilde{T}^w)_{w\in L((D_i)_{i\in I})}$. According to Proposition 4.10, it suffices to prove that Δ^{r+1} is B-recurrent. But, analogously to Proposition 3.15, can be proved that the set Δ^{r+1} is B-recurrent if for every $\varepsilon>0$, $i\in I$ and $k\in\mathbb{N}$ there exist $x,y\in\Delta^{r+1}$ and $u\in EV(\vec{w})$ with $\min dom(u) > i$ such that $d(\widetilde{T}^{u(p_1,\dots,p_m)}(y),x)<\varepsilon$ for every $1\le p_1,\dots,p_m\le k$. This finishes the proof, since $\lim_{w\in E(\vec{u})}(T_1^w\times\dots\times T_{r+1}^w)[((T_{r+1}^w)^{-1}\times\dots\times(T_{r+1}^w)^{-1})((y,\dots,y))] = (y,\dots,y)$. ☐

As a consequence of the previous theorem, via the previously defined function g, we have an analogous multiple recurrence theorem for dynamical systems indexed by an infinite semigroup with digital representation.

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