STRUCTURE OF MULTICORRELATION SEQUENCES WITH INTEGER PART POLYNOMIAL ITERATES ALONG PRIMES

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Abstract. Let T be a measure preserving $\mathbb{Z}^\ell$-action on the probability space $(X,\mathcal{B},\mu)$, $q_1,\dots,q_m\colon \mathbb{R}\to\mathbb{R}^\ell$ vector polynomials, and $f_0,\dots,f_m\in L^\infty(X)$. For any $\epsilon>0$ and multicorrelation sequences of the form $\alpha(n)=\displaystyle\int_X f_0\cdot T^{\lfloor q_1(n)\rfloor}f_1\cdots T^{\lfloor q_m(n)\rfloor}f_m\,d\mu$ we show that there exists a nilsequence ψ for which $\displaystyle\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}|\alpha(n)-\psi(n)|\le\epsilon$ and $\displaystyle\lim_{N\to\infty}\frac{1}{\pi(N)}\sum_{p\in\mathbb{P}\cap[1,N]}|\alpha(p)-\psi(p)|\le\epsilon$. This result simultaneously generalizes previous results of Frantzikinakis [2] and the authors [11, 13].

1. Introduction and main result

Since Furstenberg's ergodic theoretic proof of Szemerédi's theorem [5], there has been much interest in understanding the structure of *multicorrelation sequences*, i.e., sequences of the form

$$\alpha(n)=\int_X f_0\cdot T^n f_1\cdots T^{kn}f_k\,d\mu,\quad n\in\mathbb{N},\tag{1}$$

where $(X,\mathcal{B},\mu,T)$ is a measure preserving system and $f_0,\dots,f_k\in L^\infty(X)$. The first to provide deeper insight into the algebraic structure of such sequences were Bergelson, Host and Kra, who showed in [1] that if the system (X,μ,T) is ergodic then for any multicorrelation sequence α as in (1) there exists a uniform limit of k-step nilsequences ϕ such that

$$\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}\big|\alpha(n)-\phi(n)\big|=0.\tag{2}$$

Here, a *k-step nilsequence* is a sequence of the form $\psi(n)=F(g^n x)$, $n\in\mathbb{N}$, where F is a continuous function on a k-step nilmanifold $X=G/\Gamma$,[1] $g\in G$, $x\in X$. A *uniform limit of k-step nilsequences* is a sequence ϕ such that for every $\epsilon>0$ there exists a k-step nilsequence ψ with $\sup_{n\in\mathbb{N}}|\phi(n)-\psi(n)|\le\epsilon$.

Later, Leibman extended the result of Bergelson, Host and Kra to polynomial iterates in [14], and removed the ergodicity assumption in [15]. Another extension was obtained by the second author in [12], and independently by Tao and Teräväinen in [17], answering a question raised in [3]. There, it was shown that in addition to (2) one also has

$$\lim_{N\to\infty}\frac{1}{\pi(N)}\sum_{p\in\mathbb{P}\cap[1,N]}\big|\alpha(p)-\phi(p)\big|=0,\tag{3}$$

where $\mathbb{P}$ denotes the set of prime numbers, $[1,N]:=\{1,\dots,N\}$, and $\pi(N):=|\mathbb{P}\cap[1,N]|$.

The proofs of all the aforementioned results depend crucially on the structure theory of Host and Kra, who established in [7] that the building blocks of the factors that control

2010 *Mathematics Subject Classification.* Primary: 37A45, 37A15; Secondary: 11B30.

Key words and phrases. Multicorrelation sequences, nilsequences, integer part polynomials, prime numbers.

[1] A *k-step nilmanifold* is a homogeneous space $X=G/\Gamma$, where G a k-step nilpotent Lie group and Γ a discrete and co-compact subgroup of G.

multiple ergodic averages are nilsystems. Since the analogous factors for $\mathbb{Z}^\ell$-actions are unknown, extending the results above from $\mathbb{Z}$-actions to $\mathbb{Z}^\ell$-actions proved to be a challenge. Nevertheless, in [2] Frantzikinakis concocted a different approach and gave a description of the structure of multicorrelation sequences of $\mathbb{Z}^\ell$-actions, which we now explain.

Henceforth, let $\ell \in \mathbb{N}$ and T be a measure preserving $\mathbb{Z}^\ell$-action on a probability space $(X, \mathcal{B}, \mu)$. The system $(X, \mathcal{B}, \mu, T)$ gives rise to a more general class of correlation sequences,

$$\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m \, d\mu, \quad n \in \mathbb{N}, \tag{4}$$

where $q_1, \ldots, q_m \colon \mathbb{Z} \to \mathbb{Z}^\ell$ are integer-valued vector polynomials and $f_0, \ldots, f_m \in L^\infty(X)$. Note that (1) corresponds to the special case of (4) for $\ell = 1$ and $q_i(n) = in$. Frantzikinakis showed in [2] that for every α as in (4) and every $\epsilon > 0$ there exists a k-step nilsequence ψ such that

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \psi(n)\big| \le \epsilon, \tag{5}$$

where k only depends on ℓ, m, and the maximal degree among the polynomials $q_1, \ldots, q_m$. Moreover, in the special case where each polynomial iterate is linear, it is also proved in [2] that one can take $k = m$. It is still an open question whether in (5) one can replace ϵ with 0 after replacing the nilsequence ψ with a uniform limit of such sequences (see Question 2 in Section 3).

For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer which is smaller or equal to x, while for $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ we let $\lfloor x \rfloor \coloneqq (\lfloor x_1 \rfloor, \ldots, \lfloor x_\ell \rfloor)$. In [11], the first author extended Frantzikinakis' results to all multicorrelation sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^{\lfloor q_1(n) \rfloor} f_1 \cdots T^{\lfloor q_m(n) \rfloor} f_m \, d\mu, \quad n \in \mathbb{N}, \tag{6}$$

where $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ are real-valued vector polynomials.

More recently, the last three authors showed that the conclusion of Frantzikinakis' result also holds along the primes:

Theorem 1 ([13, Theorems A and B])**.** *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m \colon \mathbb{Z} \to \mathbb{Z}^\ell$ with degree at most d, measure preserving $\mathbb{Z}^\ell$-action T on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting α be as in* (4)*, there exists a k-step nilsequence ψ satisfying* (5) *and*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(rp+s) - \psi(rp+s)| \le \varepsilon. \tag{7}$$

In the special case $d = 1$ one can choose $k = m$.

Our main theorem simultaneously generalizes the main results from [11] and [13].

Theorem A. *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ with degree at most d, any measure preserving $\mathbb{Z}^\ell$-action T on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting α be as in* (6)*, there exists a k-step nilsequence ψ satisfying* (5) *and* (7)*. In the special case $d = 1$ one can choose $k = m$.*

The proof of Theorem A, presented in the next section, follows closely the strategy implemented in [11], but uses Theorem 1 instead of a theorem of Walsh [18] as a blackbox.

Remark 2. Both Theorems 1 and A are equivalent to seemingly stronger versions involving commuting actions. We say that two actions T_1 and T_2 of a group G commute if for every $g, h \in G$ we have $T_1^g \circ T_2^h = T_2^h \circ T_1^g$. When G is an abelian group, a collection of m commuting G-actions $T_1, \ldots, T_m$ can be identified with a single G^m-action T via $T^{(g_1,\ldots,g_m)} = T_1^{g_1} \cdots T_m^{g_m}$. Using this observation one sees that, given commuting measure preserving $\mathbb{Z}^\ell$-actions $T_1, \ldots, T_m$ in a probability space (X, μ, T), Theorem 1 holds when (4) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{q_1(n)} f_1 \cdots T_m^{q_m(n)} f_m \, d\mu, \tag{8}$$

and Theorem A holds when (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{\lfloor q_1(n) \rfloor} f_1 \cdots T_m^{\lfloor q_m(n) \rfloor} f_m \, d\mu.$$

Remark 3. Let $\lceil x \rceil$ and $[x]$ denote the smallest integer which is $\geq x$ and the closest integer to x, respectively. Using the relations $\lceil x \rceil = -\lfloor -x \rfloor$ and $[x] = \lfloor x + 1/2 \rfloor$, we see that Theorem A remains true if (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T^{[q_1(n)]_1} f_1 \cdots T^{[q_m(n)]_m} f_m \, d\mu, \quad n \in \mathbb{N},$$

where $[x]_i = ([x_1]_{i,1}, \ldots, [x_\ell]_{i,\ell})$ and $[\cdot]_{i,1}, \ldots, [\cdot]_{i,\ell}$ are any of $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$, or $[\cdot]$.

Acknowledgements. The fourth author is supported by the National Science Foundation under grant number DMS 1901453.

2. Proof of main result

We start by proving a theorem concerning flows, which stands halfway in between Theorems 1 and A. The idea behind this result is that for a real polynomial $q(x) = a_d x^d + \ldots + a_1 x + a_0 \in \mathbb{R}[x]$ and a measure presenting flow $(S^t)_{t \in \mathbb{R}}$ we can write $S^{q(n)} = (S^{a_d})^{n^d} \cdots (S^{a_0})^1$, an expression which can be handled by Theorem 1.

Theorem 4. *For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m \colon \mathbb{R} \to \mathbb{R}^\ell$ with degree at most d, commuting measure preserving $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$, and $s \in \mathbb{Z}$, letting*

$$\alpha(n) = \int_X f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m \, d\mu, \tag{9}$$

there exists a k-step nilsequence ψ satisfying (5) *and* (7)*. In the special case $d = 1$ one can choose $k = m$.*

Proof. For each $i \in [1, m]$, let $q_i = (q_{i,1}, \ldots, q_{i,\ell})$ for some $q_{i,j} \in \mathbb{R}[x]$. Next, for each $j \in [1, \ell]$, write $q_{i,j}(x) = \sum_{h=0}^d a_{i,j,h} x^h$, where the $a_{i,j,h}$'s are real numbers. Also, for each $j \in [1, \ell]$, let e_j be the j-th vector of the canonical basis of $\mathbb{R}^\ell$ and let $T_{i,j,h}$ be the measure preserving transformation defined by $T_{i,j,h} = S_i^{a_{i,j,h} e_j}$. Next, let $T_{i,h}$ be the composition $T_{i,h} = T_{i,1,h} \cdots T_{i,\ell,h}$, T_i be the $\mathbb{Z}^{d+1}$-action defined by $T_i^{(n_0,\ldots,n_d)} = T_{i,0}^{n_0} \cdots T_{i,d}^{n_d}$ and, finally, $q \colon \mathbb{Z} \to \mathbb{Z}^{d+1}$ be the polynomial $q(n) = (1, n, \ldots, n^d)$.

With this setup, for each $i \in [1, m]$ and $n \in \mathbb{N}$, we have

$$S_i^{q_i(n)} = \prod_{j=1}^{\ell} S_i^{q_{i,j}(n) e_j} = \prod_{j=1}^{\ell} \prod_{h=0}^{d} T_{i,j,h}^{n^h} = \prod_{h=0}^{d} T_{i,h}^{n^h} = T_i^{q(n)}.$$

Since the $\mathbb{R}^\ell$-actions $S_1,\dots,S_m$ commute, so do the $\mathbb{Z}^{d+1}$-actions $T_1,\dots,T_m$. This implies that the multicorrelation sequence α can be represented by an expression of the form (8). The conclusion now follows directly from Theorem 1 and Remark 2. $\square$

Next we need a result concerning the distribution of real polynomials.

Lemma 5. *Let $q\in\mathbb{R}[x]$ be a non-constant real polynomial, $r\in\mathbb{N}$ and $s\in\mathbb{Z}$. Then, denoting by $\{\cdot\}$ the fractional part, we have*

$$\lim_{\delta\to0^+}\lim_{N-M\to\infty}\frac{1}{N-M}\left|\Big\{n\in[M,N):\big\{q(n)\big\}\in[1-\delta,1)\Big\}\right|=0,$$

and

$$\lim_{\delta\to0^+}\lim_{N\to\infty}\frac{1}{\pi(N)}\left|\Big\{p\in\mathbb{P}\cap[1,N]:\big\{q(rp+s)\big\}\in[1-\delta,1)\Big\}\right|=0.$$

Proof. Let

$$A(\delta)=\lim_{N-M\to\infty}\frac{1}{N-M}\left|\Big\{n\in[M,N):\big\{q(n)\big\}\in[1-\delta,1)\Big\}\right|,$$

and

$$B(\delta)=\lim_{N\to\infty}\frac{1}{\pi(N)}\left|\Big\{p\in\mathbb{P}\cap[1,N]:\big\{q(rp+s)\big\}\in[1-\delta,1)\Big\}\right|.$$

If $q-q(0)$ has an irrational coefficient, then by Weyl's Uniform Distribution Theorem [19] and Rhin's Theorem [16] we have $A(\delta)=B(\delta)=\delta$ which approach 0 as $\delta\to0^+$.

Assume otherwise that $q\in\mathbb{R}[x]$ satisfies $q-q(0)\in\mathbb{Q}[x]$, say $q(x)=q(0)+b^{-1}\sum_{j=1}^{\ell}a_jx^j$ where $b\in\mathbb{N}$, $a_j\in\mathbb{Z}$ for $1\le j\le\ell$, and $q(0)\in\mathbb{R}$. It follows that for all $n\in\mathbb{N}$,

$$q(n)-q(0)\bmod 1\in\left\{0,\frac{1}{b},\frac{2}{b},\dots,\frac{b-1}{b}\right\}.$$

In particular, the fractional part $\{q(n)\}$ takes only finitely many values. Therefore, if δ is small enough, for every $n\in\mathbb{N}$ we have $\{q(n)\}\notin[1-\delta,1)$ and hence $A(\delta)=B(\delta)=0$, which implies the desired conclusion. $\square$

For the proof of Theorem A we adapt arguments from [10, 11], i.e., we use a multidimensional suspension flow to approximate α by a multicorrelation sequence of the form (9). The arising error consists of terms of the form $1_{\{n\in\mathbb{N}:\,q(n)\in[1-\delta,1)\}}$ that can be controlled by Lemma 5.

Proof of Theorem A. Given $\ell,m,d\in\mathbb{N}$, let k be as guaranteed by Theorem 4. Let $q_1,\dots,q_m$, T, $f_0,\dots,f_m$, $\epsilon>0$, $r\in\mathbb{N}$, $s\in\mathbb{Z}$ and α be as in the statement. By multiplying each function by a constant if needed, we can assume without loss of generality that $\|f_i\|_\infty\le1$ for each $i\in[1,m]$.

We start by considering a multidimensional suspension flow with a constant 1 ceiling function. More precisely, let $Y:=X\times[0,1)^{m\times\ell}$ and $\nu=\mu\otimes\lambda$, where λ denotes the Lebesgue measure on $[0,1)^{m\times\ell}$. For each $i\in[1,m]$ define the measure preserving $\mathbb{R}^\ell$-action S_i on (Y,ν) as follows: for any $t\in\mathbb{R}^\ell$ and $(x;b_1,\dots,b_m)\in Y=X\times\big([0,1)^\ell\big)^m$, let

$$S_i^t(x;b_1,\dots,b_m):=\big(T^{\lfloor b_i+t\rfloor}x;b_1,\dots,b_{i-1},\{b_i+t\},b_{i+1},\dots,b_m\big),$$

where $\{u\}:=u-\lfloor u\rfloor$ for any $u\in\mathbb{R}^\ell$. Observe that the actions $S_1,\dots,S_m$ commute.

Let $\pi\colon Y\to X$ be the natural projection and $\delta>0$ a small parameter to be determined later. For each $i\in[1,m]$ let $\hat f_i\in L^\infty(Y)$ be the composition $\hat f_i:=f_i\circ\pi$, and $\hat f_0:=1_{X\times[0,\delta]^{m\times\ell}}\cdot f_0\circ\pi$. Define

$$\tilde\alpha(n)=\int_Y\hat f_0\cdot S_1^{q_1(n)}\hat f_1\cdots S_m^{q_m(n)}\hat f_m\,d\nu.$$

By Theorem 4 there exists a k-step nilsequence $\tilde{\psi}$ such that

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N-1} |\tilde{\alpha}(n)-\tilde{\psi}(n)| \le \delta^{\ell m}\epsilon/2, \tag{10}$$

and

$$\lim_{N\to\infty} \frac{1}{\pi(N)}\sum_{p\in\mathbb{P}\cap[1,N]} |\tilde{\alpha}(rp+s)-\tilde{\psi}(rp+s)| \le \delta^{\ell m}\epsilon/2. \tag{11}$$

On the other hand,

$$\tilde{\alpha}(n) = \int_{[0,\delta]^{\ell m}}\int_X f_0(x) f_1\Big(T^{\lfloor q_1(n)+b_1\rfloor}x\Big)\cdots f_m\Big(T^{\lfloor q_m(n)+b_m\rfloor}x\Big)\, d\mu(x)\, d\lambda(b_1,\ldots,b_m),$$

which implies

$$\alpha(n)-\frac{\tilde{\alpha}(n)}{\delta^{\ell m}} = \frac{1}{\delta^{\ell m}}\int_{[0,\delta]^{\ell m}}\int_X f_0(x)\left(\prod_{i=1}^m f_i\Big(T^{\lfloor q_i(n)\rfloor}x\Big)-\prod_{i=1}^m f_i\Big(T^{\lfloor q_i(n)+b_i\rfloor}x\Big)\right) d\mu\, d\lambda. \tag{12}$$

In particular, it follows from (12) that $|\alpha(n)-\delta^{-\ell m}\tilde{\alpha}(n)|\le 2$ for all $n\in\mathbb{N}$. If $b_i\in[0,\delta]^\ell$ and $\big\{q_i(n)\big\}\in[0,1-\delta)^\ell$ then $\lfloor q_i(n)+b_i\rfloor=\lfloor q_i(n)\rfloor$. Therefore (12) also implies that $\alpha(n)=\delta^{-\ell m}\tilde{\alpha}(n)$ whenever

$$n\notin\Big\{n\in\mathbb{N}:\big\{q_i(n)\big\}\in[1-\delta,1)^\ell \text{ for some } i\in[1,m]\Big\}.$$

In view of Lemma 5, by choosing δ small enough, we have

$$\lim_{N-M\to\infty} \frac{1}{N-M}\sum_{n=M}^{N-1} \Big|\alpha(n)-\delta^{-\ell m}\tilde{\alpha}(n)\Big| < \frac{\epsilon}{2} \tag{13}$$

and

$$\lim_{N\to\infty} \frac{1}{\pi(N)}\sum_{p\in\mathbb{P}\cap[1,N]} \Big|\alpha(rp+s)-\delta^{-\ell m}\tilde{\alpha}(rp+s)\Big| < \frac{\epsilon}{2}. \tag{14}$$

Letting $\psi=\delta^{-m\ell}\tilde{\psi}$ and combining (10) with (13) and (11) with (14) we obtain the desired conclusion. □

Remark 6. As it was already mentioned in Section 1, it is an open problem whether one can improve upon the approximation in Frantzikinakis' main result in [2] and Theorem 1 and take $\epsilon=0$ in (5) and (7) (see Question 2 below). However, as the following example shows, in the case of Theorem A it is not possible to improve upon the approximation in that manner.

Example 7. Take $X=\mathbb{T}:=\mathbb{R}/\mathbb{Z}$, $T(x)=x+1/\sqrt{2}$, $q(n)=\sqrt{2}n$, $f_0(x)=e(x)$ and $f_1(x)=e(-x)$, where $e(x):=e^{2\pi i x}$. Then we have

$$\begin{aligned}\alpha(n) &= \int f_0\cdot T^{\lfloor q(n)\rfloor}f_1\, d\mu = \int e(x)e\left(-x-\frac{1}{\sqrt{2}}\lfloor\sqrt{2}n\rfloor\right)dx\\ &= e\left(-\frac{1}{\sqrt{2}}\lfloor\sqrt{2}n\rfloor\right) = e\left(\frac{1}{\sqrt{2}}\{\sqrt{2}n\}\right).\end{aligned}$$

In particular, we can write $\alpha(n)$ as $F(T^n x_0)$ with $x_0=0\in\mathbb{T}$ and $F(x)=e(\{x\}/\sqrt{2})$ for $x\in\mathbb{T}$. Assume for the sake of a contradiction that there exists a uniform limit of nilsequences ϕ for which

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^N |\alpha(n)-\phi(n)| = 0. \tag{15}$$

By [9, Lemma 18], ϕ can be written as $\phi(n) = G(S^n y_0)$ for all $n \in \mathbb{N}$, where G is a continuous function on an inverse limit of nilsystems (Y, S) and $y_0 \in Y$.

We claim that $\alpha(n) = \phi(n)$ for all $n \in \mathbb{N}$. If not, then there exists $\delta > 0$ and $n_0 \in \mathbb{N}$ such that

$$|\alpha(n_0) - \phi(n_0)| = |F(T^{n_0} x_0) - G(S^{n_0} y_0)| \geq \delta. \tag{16}$$

Since the system $(X \times Y, T \times S)$ is the product of two distal systems, is a distal system itself. This implies that the point $(T^{n_0} x_0, S^{n_0} y_0)$ is uniformly recurrent, i.e., the sequence $(T^n x_0, S^n y_0)$ visits any neighborhood of $(T^{n_0} x_0, S^{n_0} y_0)$ in a syndetic set. This fact together with (16) and the fact that both the real and imaginary parts of F are almost everywhere continuous and semicontinuous imply that the set

$$\{n \in \mathbb{N} : |F(T^n x_0) - G(S^n y_0)| \geq \delta/2\}$$

is syndetic, which contradicts (15). Hence $\alpha(n) = \phi(n)$ for all $n \in \mathbb{N}$. However, by [6, Proposition 4.2.5], the sequence α is not a distal sequence; in particular, it is not a uniform limit of nilsequences, contradicting our assumption.

3. Open questions

We close this article with three open questions. Theorem A provides an approximation result of multicorrelation sequences along an integer polynomial of degree one, evaluated at primes. We can ask whether a similar result is true along other classes of sequences.

Question 1. *Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $c > 0$, and p_n denote the n-th prime. Suppose $r_n = q(n), q(p_n), \lfloor n^c \rfloor$ or $\lfloor p_n^c \rfloor$ for $n \in \mathbb{N}$. Is it true that for any α as in* (6) *and $\epsilon > 0$, there exists a nilsequence ψ satisfying*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(r_n) - \psi(r_n)| \leq \epsilon?$$

Variants of the following question have appeared several times in the literature, e.g., [2, Remark after Theorem 1.1], [3, Problem 20], [4, Problem 1], and [8, Page 398].

Question 2. *Let α be as in* (4)*. Does there exist a uniform limit of nilsequences ϕ such that*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \phi(n)\big| = 0?$$

As mentioned in Example 7, the answer to Question 2 is negative when α is a multicorrelation sequence as in (6). Nevertheless, it makes sense to ask for the following modification of it.

Question 3. *Let α be as in* (6)*. Does there exist a uniform limit of* Riemann integrable nilsequences *ϕ satisfying*

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} \big|\alpha(n) - \phi(n)\big| = 0?$$

Here we say that ϕ is a uniform limit of Riemann integrable nilsequences if for every $\epsilon > 0$ there exists a nilmanifold $X = G/\Gamma$, a point $x \in X$, $g \in G$ and a Riemann integrable function[2] *$F\colon X \to \mathbb{C}$ such that $\sup_{n\in\mathbb{N}} |\phi(n) - F(g^n x)| < \epsilon$.*

[2] A function F is *Riemann integrable* on a nilmanifold if its points of discontinuity is a null set with respect to the Haar measure.

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