STRUCTURE OF MULTICORRELATION SEQUENCES WITH INTEGER PART POLYNOMIAL ITERATES ALONG PRIMES

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ABSTRACT. Let $T$ be a measure preserving $\mathbb{Z}^d$-action on the probability space $(X, \mathcal{B}, \mu)$, $q_1, \ldots, q_m : \mathbb{R} \to \mathbb{R}^d$ vector polynomials, and $f_0, \ldots, f_m \in L^\infty(X)$. For any $\epsilon > 0$ and multicorrelation sequences of the form $\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_m \, d\mu$ we show that there exists a nilsequence $\psi$ for which

$$\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| \leq \epsilon$$

and

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(p) - \psi(p)| \leq \epsilon.$$ 

This result simultaneously generalizes previous results of Frantzikinakis [2] and the authors [11, 13].

1. Introduction and main result

Since Furstenberg’s ergodic theoretic proof of Szemerédi’s theorem [3], there has been much interest in understanding the structure of multicorrelation sequences, i.e., sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^{q_1(n)} f_1 \cdots T^{q_m(n)} f_k \, d\mu, \quad n \in \mathbb{N},$$

where $(X, \mathcal{B}, \mu, T)$ is a measure preserving system and $f_0, \ldots, f_k \in L^\infty(X)$. The first to provide deeper insight into the algebraic structure of such sequences were Bergelson, Host and Kra, who showed in [11] that if the system $(X, \mu, T)$ is ergodic then for any multicorrelation sequence $\alpha$ as in (1) there exists a uniform limit of $k$-step nilsequences $\psi$ such that

$$\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| = 0.$$  

Here, a $k$-step nilsequence is a sequence of the form $\psi(n) = F(g^n x)$, $n \in \mathbb{N}$, where $F$ is a continuous function on a $k$-step nilmanifold $X = G/\Gamma$, $g \in G$, $x \in X$. A uniform limit of $k$-step nilsequences is a sequence $\psi$ such that for every $\epsilon > 0$ there exists a $k$-step nilsequence $\psi$ with $\sup_{n \in \mathbb{N}} |\psi(n) - \psi(p)| \leq \epsilon$.

Later, Leibman extended the result of Bergelson, Host and Kra to polynomial iterates in [14], and removed the ergodicity assumption in [15]. Another extension was obtained by the second author in [12], and independently by Tao and Teräväinen in [17], answering a question raised in [3]. There, it was shown that in addition to (2) one also has

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(p) - \psi(p)| = 0,$$

where $\mathbb{P}$ denotes the set of prime numbers, $[1, N] := \{1, \ldots, N\}$, and $\pi(N) := |\mathbb{P} \cap [1, N]|$.

The proofs of all the aforementioned results depend crucially on the structure theory of Host and Kra, who established in [7] that the building blocks of the factors that control

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1A $k$-step nilmanifold is a homogeneous space $X = G/\Gamma$, where $G$ a $k$-step nilpotent Lie group and $\Gamma$ a discrete and co-compact subgroup of $G$. 

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multiple ergodic averages are nilsystems. Since the analogous factors for $\mathbb{Z}^\ell$-actions are unknown, extending the results above from $\mathbb{Z}$-actions to $\mathbb{Z}^\ell$-actions proved to be a challenge. Nevertheless, in [2] Frantzikinakis concocted a different approach and gave a description of the structure of multicorrelation sequences of $\mathbb{Z}^\ell$-actions, which we now explain.

Henceforth, let $\ell \in \mathbb{N}$ and $T$ be a measure preserving $\mathbb{Z}^\ell$-action on a probability space $(X, \mathcal{B}, \mu)$. The system $(X, \mathcal{B}, \mu, T)$ gives rise to a more general class of correlation sequences,

$$\alpha(n) = \int_X f_0 \cdot T^{n_1(n)} f_1 \cdots T^{n_m(n)} f_m \, d\mu, \quad n \in \mathbb{N},$$

where $q_1, \ldots, q_m : \mathbb{Z} \to \mathbb{Z}^\ell$ are integer-valued vector polynomials and $f_0, \ldots, f_m \in L^\infty(X)$. Note that (1) corresponds to the special case of (4) for $\ell = 1$ and $q_i(n) = i$. Frantzikinakis showed in [2] that for every $\alpha$ as in (4) and every $\varepsilon > 0$ there exists a $k$-step nilsequence $\psi$ such that

$$\lim_{N \to \infty} \frac{1}{N} \sum_{n=M}^{N-1} |\alpha(n) - \psi(n)| \leq \varepsilon,$$

where $k$ only depends on $\ell$, $m$, and the maximal degree among the polynomials $q_1, \ldots, q_m$. Moreover, in the special case where each polynomial iterate is linear, it is also proved in [2] that one can take $k = m$. It is still an open question whether in (5) one can replace $\varepsilon$ with 0 after replacing the nilsequence $\psi$ with a uniform limit of such sequences (see Question 2 in Section 3).

For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ the largest integer which is smaller or equal to $x$, while for $x = (x_1, \ldots, x_\ell) \in \mathbb{R}^\ell$ we let $\lfloor x \rfloor := (\lfloor x_1 \rfloor, \ldots, \lfloor x_\ell \rfloor)$. In [11], the first author extended Frantzikinakis’ results to all multicorrelation sequences of the form

$$\alpha(n) = \int_X f_0 \cdot T^{[q_1(n)]} f_1 \cdots T^{[q_m(n)]} f_m \, d\mu, \quad n \in \mathbb{N},$$

where $q_1, \ldots, q_m : \mathbb{R} \to \mathbb{R}^\ell$ are real-valued vector polynomials.

More recently, the last three authors showed that the conclusion of Frantzikinakis’ result also holds along the primes:

**Theorem 1** ([13] Theorems A and B). For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m : \mathbb{Z} \to \mathbb{Z}^\ell$ with degree at most $d$, measure preserving $\mathbb{Z}^\ell$-action $T$ on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting $\alpha$ be as in (4), there exists a $k$-step nilsequence $\psi$ satisfying (5) and

$$\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} |\alpha(rp + s) - \psi(rp + s)| \leq \varepsilon.$$

In the special case $d = 1$ one can choose $k = m$.

Our main theorem simultaneously generalizes the main results from [11] and [13].

**Theorem A.** For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m : \mathbb{R} \to \mathbb{R}^\ell$ with degree at most $d$, any measure preserving $\mathbb{Z}^\ell$-action $T$ on a probability space $(X, \mathcal{B}, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$, letting $\alpha$ be as in (6), there exists a $k$-step nilsequence $\psi$ satisfying (5) and (7). In the special case $d = 1$ one can choose $k = m$.

The proof of Theorem A presented in the next section, follows closely the strategy implemented in [11], but uses Theorem 1 instead of a theorem of Walsh [13] as a blackbox.
Remark 2. Both Theorems [1] and [A] are equivalent to seemingly stronger versions involving commuting actions. We say that two actions $T_1$ and $T_2$ of a group $G$ commute if for every $g, h \in G$ we have $T_1^g \circ T_2^h = T_2^h \circ T_1^g$. When $G$ is an abelian group, a collection of $m$ commuting $G$-actions $T_1, \ldots, T_m$ can be identified with a single $G^m$-action $T$ via $T^{(g_1, \ldots, g_m)} = T_{g_1} \circ \cdots \circ T_{g_m}$. Using this observation one sees that, given commuting measure-preserving $Z^d$-actions $T_1, \ldots, T_m$ in a probability space $(X, \mu, T)$, Theorem [1] holds when (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{\ell_0(n)} f_1 \cdots T_m^{\ell_m(n)} f_m \, d\mu,$$

and Theorem [A] holds when (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{[q_1(n)]} f_1 \cdots T_m^{[q_m(n)]} f_m \, d\mu.$$

Remark 3. Let $\lfloor x \rfloor$ and $\lceil x \rceil$ denote the smallest integer which is $\leq x$ and the closest integer to $x$, respectively. Using the relations $\lfloor x \rfloor = -\lceil -x \rceil$ and $\lfloor x + 1/2 \rfloor$, we see that Theorem [A] remains true if (6) is replaced by

$$\alpha(n) = \int_X f_0 \cdot T_1^{\lfloor q_1(n) \rfloor} f_1 \cdots T_m^{\lfloor q_m(n) \rfloor} f_m \, d\mu, \quad n \in \mathbb{N},$$

where $\lfloor x \rfloor_i = ([x]_i, 1, \ldots, [x]_{\ell_i, i})$ and $\lceil \cdot \rceil_i, 1, \ldots, \lceil \cdot \rceil_{i, \ell}$ are any of $\lfloor \cdot \rfloor$, $\lceil \cdot \rceil$, or $\lfloor \cdot \rfloor$.

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2. PROOF OF MAIN RESULT

We start by proving a theorem concerning flows, which stands halfway in between Theorems [1] and [A]. The idea behind this result is that for a real polynomial $q(x) = a_dx^d + \cdots + a_1x + a_0 \in \mathbb{R}[x]$ and a measure-presenting flow $(S^t)_{t \in \mathbb{R}}$ we can write $S^{q(n)} = (S^{a_d})^{n_d} \cdots (S^{a_0})^{n_0}$, an expression which can be handled by Theorem [1].

**Theorem 4.** For every $\ell, m, d \in \mathbb{N}$ there exists $k \in \mathbb{N}$ with the following property. For any polynomials $q_1, \ldots, q_m : \mathbb{R} \to \mathbb{R}^\ell$ with degree at most $d$, commuting measure-preserving $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ on a probability space $(X, B, \mu)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(X)$, $\varepsilon > 0$, $r \in \mathbb{N}$, and $s \in \mathbb{Z}$, letting

$$\alpha(n) = \int_X f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m \, d\mu,$$

there exists a $k$-step nilsequence $\psi$ satisfying (5) and (7). In the special case $d = 1$ one can choose $k = m$.

**Proof.** For each $i \in [1, m]$, let $q_i = (q_{i, 1}, \ldots, q_{i, \ell})$ for some $q_{i,j} \in \mathbb{R}[x]$. Next, for each $j \in [1, \ell]$, write $q_{i,j}(x) = \sum_{h=0}^{d} a_{i,j,h}x^h$, where the $a_{i,j,h}$’s are real numbers. Also, for each $j \in [1, \ell]$, let $e_j$ be the $j$-th vector of the canonical basis of $\mathbb{R}^\ell$ and let $T_{i,j,h}$ be the measure-preserving transformation defined by $T_{i,j,h} = S_i^{a_{i,j,h}e_j}$. Next, let $T_{i,h}$ be the composition

$$T_{i,h} = T_{i,1,h} \cdots T_{i,\ell,h},$$

be the $Z^{d+1}$-action defined by $T_{i}^{(n_0, \ldots, n_d)} = T_{i,0}^{n_0} \cdots T_{i,d}^{n_d}$ and, finally, $q : Z \to Z^{d+1}$ be the polynomial $q(n) = (1, n, \ldots, n^d)$.

With this setup, for each $i \in [1, m]$ and $n \in \mathbb{N}$, we have

$$S_i^{q_i(n)} = \prod_{j=1}^{\ell} S_i^{q_{i,j}(n)e_j} = \prod_{j=1}^{\ell} \prod_{h=0}^{d} T_{i,j,h}^{h} = \prod_{h=0}^{d} T_{i,h}^{h} = T_i^{q(n)}.$$
Since the $\mathbb{R}^\ell$-actions $S_1, \ldots, S_m$ commute, so do the $\mathbb{Z}^{d+1}$-actions $T_1, \ldots, T_m$. This implies that the multicorrelation sequence $\alpha$ can be represented by an expression of the form $[9]$. The conclusion now follows directly from Theorem $[1]$ and Remark $[2]$. □

Next we need a result concerning the distribution of real polynomials.

**Lemma 5.** Let $q \in \mathbb{R}[x]$ be a non-constant real polynomial, $r \in \mathbb{N}$ and $s \in \mathbb{Z}$. Then, denoting by $\{ \cdot \}$ the fractional part, we have

$$\lim_{\delta \to 0^+ \atop N \to \infty} \frac{1}{N-M} \left| \left\{ n \in [M,N) : \{ q(n) \} \in [1-\delta, 1) \right\} \right| = 0,$$

and

$$\lim_{\delta \to 0^+ \atop N \to \infty} \frac{1}{\pi(N)} \left| \left\{ p \in \mathbb{P} \cap [1,N] : \{ q(rp+s) \} \in [1-\delta, 1) \right\} \right| = 0.$$

**Proof.** Let

$$A(\delta) = \lim_{N \to \infty \atop M \to \infty} \frac{1}{N-M} \left| \left\{ n \in [M,N) : \{ q(n) \} \in [1-\delta, 1) \right\} \right|,$$

and

$$B(\delta) = \lim_{N \to \infty \atop M \to \infty} \frac{1}{\pi(N)} \left| \left\{ p \in \mathbb{P} \cap [1,N] : \{ q(rp+s) \} \in [1-\delta, 1) \right\} \right|.$$

If $q - q(0)$ has an irrational coefficient, then by Weyl’s Uniform Distribution Theorem $[19]$ and Rhin’s Theorem $[16]$ we have $A(\delta) = B(\delta) = \delta$ which approach 0 as $\delta \to 0^+$. Assume otherwise that $q \in \mathbb{R}[x]$ satisfies $q - q(0) \in \mathbb{Q}[x]$, say $q(x) = q(0) + b^{-1} \sum_{j=1}^\ell a_j x^j$ where $b \in \mathbb{N}$, $a_j \in \mathbb{Z}$ for $1 \leq j \leq \ell$, and $q(0) \in \mathbb{R}$. It follows that for all $n \in \mathbb{N}$,

$$q(n) - q(0) \text{ mod } 1 \in \left\{ 0, \frac{1}{b}, \frac{2}{b}, \ldots, \frac{b-1}{b} \right\}.$$

In particular, the fractional part $\{ q(n) \}$ takes only finitely many values. Therefore, if $\delta$ is small enough, for every $n \in \mathbb{N}$ we have $\{ q(n) \} \notin [1-\delta, 1)$ and hence $A(\delta) = B(\delta) = 0$, which implies the desired conclusion. □

For the proof of Theorem $[A]$ we adapt arguments from $[10][11]$, i.e., we use a multidimensional suspension flow to approximate $\alpha$ by a multicorrelation sequence of the form $[9]$. The arising error consists of terms of the form $1_{\{n \in \mathbb{N} : q(n) \notin [1-\delta, 1)\}}$ that can be controlled by Lemma $[5]$.

**Proof of Theorem $[A]$** Given $\ell, m, d \in \mathbb{N}$, let $k$ be as guaranteed by Theorem $[4]$. Let $q_1, \ldots, q_m, T_1, f_0, \ldots, f_m, \epsilon > 0, r \in \mathbb{N}$, $s \in \mathbb{Z}$ and $\alpha$ be as in the statement. By multiplying each function by a constant if needed, we can assume without loss of generality that $\|f_i\|_\infty \leq 1$ for each $i \in [1,m]$.

We start by considering a multidimensional suspension flow with a constant 1 ceiling function. More precisely, let $Y := X \times [0,1)^{m \times \ell}$ and $\nu = \mu \otimes \lambda$, where $\lambda$ denotes the Lebesgue measure on $[0,1)^{m \times \ell}$. For each $i \in [1,m]$ define the measure preserving $\mathbb{R}^\ell$-action $S_i$ on $(Y, \nu)$ as follows: for any $x \in \mathbb{R}^\ell$ and $(x;b_1,\ldots,b_m) \in Y = X \times (0,1)^{m \times \ell}$, let

$$S_i^t(x;b_1,\ldots,b_m) := (T^{[b_i+\epsilon]} x;b_1,\ldots,b_{i-1},b_i+t,b_{i+1},\ldots,b_m),$$

where $\{u\} := u - \lfloor u \rfloor$ for any $u \in \mathbb{R}\ell$. Observe that the actions $S_1, \ldots, S_m$ commute.

Let $\pi : Y \to X$ be the natural projection and $\delta > 0$ a small parameter to be determined later. For each $i \in [1,m]$ let $f_i \in L^\infty(Y)$ be the composition $f_i := f_i \circ \pi$, and $f_0 := 1_{X \times [0,\delta^{m \times \ell}] \times \ell} \cdot f_0 \circ \pi$. Define

$$\tilde{\alpha}(n) = \int_Y f_0 \cdot S_1^{q_1(n)} f_1 \cdots S_m^{q_m(n)} f_m \, d\nu.$$
By Theorem 4 there exists a $k$-step nilsequence $\hat{\psi}$ such that
\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\hat{\alpha}(n) - \hat{\psi}(n)| \leq \delta^{1/m} \epsilon/2, \tag{10}
\]
and
\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathcal{P} \cap [1,N]} |\hat{\alpha}(rp + s) - \hat{\psi}(rp + s)| \leq \delta^{1/m} \epsilon/2. \tag{11}
\]
On the other hand,
\[
\hat{\alpha}(n) = \int_{[0,\delta^{1/m}]} \int_X f_0(x) f_1 \left(T^{[q(n)+b_1]} x\right) \cdots f_m \left(T^{[q_m(n)+b_m]} x\right) \, d\mu(x) \, d\lambda(b_1, \ldots, b_m),
\]
which implies
\[
\alpha(n) - \hat{\alpha}(n) = \frac{1}{\delta^{1/m}} \int_{[0,\delta^{1/m}]} \int_X f_0(x) \left( \prod_{i=1}^m \left( T^{[q_i(n)]} x \right) - \prod_{i=1}^m \left( T^{[q_i(n)+b_i]} x \right) \right) \, d\mu \, d\lambda.
\]
In particular, it follows from (12) that $|\alpha(n) - \delta^{-1/m} \hat{\alpha}(n)| \leq 2$ for all $n \in \mathbb{N}$. If $b_i \in [0, \delta)^\ell$ and $\{q_i(n)\} \in [0, 1-\delta)^\ell$ then $[q_i(n) + b_i] = [q_i(n)]$. Therefore (12) also implies that $\alpha(n) = \delta^{-1/m} \hat{\alpha}(n)$ whenever
\[
n \not\in \left\{ n \in \mathbb{N} : \{q_i(n)\} \in [1-\delta, 1)^\ell \text{ for some } i \in \{1, m\} \right\}.
\]
In view of Lemma 3 by choosing $\delta$ small enough, we have
\[
\lim_{N \to \infty} \sum_{n=M}^{N-1} |\alpha(n) - \delta^{-1/m} \hat{\alpha}(n)| < \frac{\epsilon}{2} \tag{13}
\]
and
\[
\lim_{N \to \infty} \frac{1}{\pi(N)} \sum_{p \in \mathcal{P} \cap [1,N]} |\alpha(rp + s) - \delta^{-1/m} \hat{\alpha}(rp + s)| < \frac{\epsilon}{2}. \tag{14}
\]
Letting $\psi = \delta^{-1/m} \hat{\psi}$ and combining (10) with (13) and (11) with (14) we obtain the desired conclusion.

Remark 6. As it was already mentioned in Section 1 it is an open problem whether one can improve upon the approximation in Frantzikinakis’ main result in [2] and Theorem 1 and take $\epsilon = 0$ in [3] and [7] (see Question 2 below). However, as the following example shows, in the case of Theorem X it is not possible to improve upon the approximation in that manner.

Example 7. Take $X = T := \mathbb{R}/\mathbb{Z}$, $T(x) = x + 1/\sqrt{2}$, $q(n) = \sqrt{2}n$, $f_0(x) = e(x)$ and $f_1(x) = e(-x)$, where $e(x) := e^{2\pi i x}$. Then we have
\[
\alpha(n) = \int f_0 \cdot T^{[q(n)]} f_1 \, d\mu = \int e(x) e \left( -x - \frac{1}{\sqrt{2}} \lfloor \sqrt{2}n \rfloor \right) \, dx = e \left( -\frac{1}{\sqrt{2}} \lfloor \sqrt{2}n \rfloor \right) = e \left( \frac{1}{\sqrt{2}} \{\sqrt{2}n\} \right).
\]
In particular, we can write $\alpha(n)$ as $F(T^n x_0)$ with $x_0 = 0 \in T$ and $F(x) = e(\{x\}/\sqrt{2})$ for $x \in T$. Assume for the sake of a contradiction that there exists a uniform limit of nilsequences $\phi$ for which
\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(n) - \phi(n)| = 0. \tag{15}
\]
By [9, Lemma 18], \( \phi \) can be written as \( \phi(n) = G(S^n y_0) \) for all \( n \in \mathbb{N} \), where \( G \) is a continuous function on an inverse limit of nilsystems \((Y,S)\) and \( y_0 \in Y \).

We claim that \( \alpha(n) = \phi(n) \) for all \( n \in \mathbb{N} \). If not, then there exists \( \delta > 0 \) and \( n_0 \in \mathbb{N} \) such that

\[
|\alpha(n_0) - \phi(n_0)| = |F(T^{n_0}x_0) - G(S^{n_0}y_0)| \geq \delta. \tag{16}
\]

Since the system \((X \times Y, T \times S)\) is the product of two distal systems, is a distal system itself. This implies that the point \((T^{n_0}x_0, S^{n_0}y_0)\) is uniformly recurrent, i.e., the sequence \((T^{n_0}x_0, S^{n_0}y_0)\) visits any neighborhood of \((T^{n_0}x_0, S^{n_0}y_0)\) in a syndetic set. This fact together with \((16)\) and the fact that both the real and imaginary parts of \( F \) are almost everywhere continuous and semicontinuous imply that the set

\[
\{ n \in \mathbb{N} : |F(T^n x_0) - G(S^n y_0)| \geq \delta/2 \}
\]

is syndetic, which contradicts \([15]\). Hence \( \alpha(n) = \phi(n) \) for all \( n \in \mathbb{N} \). However, by [6, Proposition 4.2.5], the sequence \( \alpha \) is not a distal sequence; in particular, it is not a uniform limit of nilsequences, contradicting our assumption.

### 3. Open Questions

We close this article with three open questions. Theorem [A] provides an approximation result of multicorrelation sequences along an integer polynomial of degree one, evaluated at primes. We can ask whether a similar result is true along other classes of sequences.

**Question 1.** Let \( q \in \mathbb{R}[x] \) be a non-constant real polynomial, \( c > 0 \), and \( p_n \) denote the \( n \)-th prime. Suppose \( r_n = q(n), q(p_n), \lfloor n^c \rfloor \) or \( \lceil p_n^c \rceil \) for \( n \in \mathbb{N} \). Is it true that for any \( \alpha \) as in \((6)\) and \( \epsilon > 0 \), there exists a nilsequence \( \psi \) satisfying

\[
\lim_{N \to \infty} \frac{1}{N} \sum_{n=1}^{N} |\alpha(r_n) - \psi(r_n)| \leq \epsilon?
\]

Variants of the following question have appeared several times in the literature, e.g., [2, Remark after Theorem 1.1], [3, Problem 20], [4, Problem 1], and [8, Page 398].

**Question 2.** Let \( \alpha \) be as in \((6)\). Does there exist a uniform limit of nilsequences \( \phi \) such that

\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0?
\]

As mentioned in Example [7], the answer to Question 2 is negative when \( \alpha \) is a multicorrelation sequence as in \((6)\). Nevertheless, it makes sense to ask for the following modification of it.

**Question 3.** Let \( \alpha \) be as in \((6)\). Does there exist a uniform limit of Riemann integrable nilsequences \( \phi \) satisfying

\[
\lim_{N \to \infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |\alpha(n) - \phi(n)| = 0?
\]

Here we say that \( \phi \) is a uniform limit of Riemann integrable nilsequences if for every \( \epsilon > 0 \) there exists a nilmanifold \( X = G/\Gamma \), a point \( x \in X \), \( g \in G \) and a Riemann integrable function \( F : X \to \mathbb{C} \) such that \( \sup_{n \in \mathbb{N}} |\phi(n) - F(g^n x)| < \epsilon \).

\[^2\text{A function } F \text{ is Riemann integrable on a nilmanifold if its points of discontinuity is a null set with respect to the Haar measure.} \]
References


