

MULTIPLICATIVE COMBINATORIAL PROPERTIES OF RETURN TIME SETS IN MINIMAL DYNAMICAL SYSTEMS

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ABSTRACT. We investigate the relationship between the dynamical properties of minimal topological dynamical systems and the multiplicative combinatorial properties of return time sets arising from those systems. In particular, we prove that for a residual set of points in any minimal system, the set of return times to any non-empty, open set contains arbitrarily long geometric progressions. Under the separate assumptions of total minimality and distality, we prove that return time sets have positive multiplicative upper Banach density along \mathbb{N} and along cosets of multiplicative subsemigroups of \mathbb{N} , respectively. The primary motivation for this work is the long-standing open question of whether or not syndetic subsets of the positive integers contain arbitrarily long geometric progressions; our main result is some evidence for an affirmative answer to this question.

1. INTRODUCTION

1.1. Results. Let $T : X \rightarrow X$ be a continuous map of a compact metric space (X, d) . In the topological dynamical system (X, T) , the *set of return times* of a point $x \in X$ to a non-empty, open set $U \subseteq X$ is

$$R(x, U) := \{n \in \mathbb{N} \mid T^n x \in U\}.$$

Much is known about the relationship between dynamical properties of the system (X, T) and the additive combinatorial properties of the sets $R(x, U)$. For example, if (X, T) is *minimal* (that is, for all $x \in X$, the set $\{T^n x \mid n \in \mathbb{N}\}$ is dense in X), then every set of return times $R(x, U)$ is *syndetic*, meaning that there exists $N \in \mathbb{N}$ such that $R(x, U)$ has non-empty intersection with every interval of N consecutive positive integers. This connection between dynamics and additive combinatorics has had a strong influence in Ramsey Theory; we discuss some of the history behind this connection and put our main results into context in Section 1.2.

In this paper, we consider the relationship between the dynamical properties of the system (X, T) and the *multiplicative* combinatorial properties of the sets $R(x, U)$. Our first main result concerns geometric progressions, configurations of the form $\{nm, nm^2, \dots, nm^\ell\}$, in sets of return times in minimal systems.

Theorem 1.1. *Let (X, T) be a minimal dynamical system. There exists a residual set $X' \subseteq X$ such that for all $x \in X'$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ contains arbitrarily long geometric progressions.*

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We are able to strengthen the conclusion of Theorem 1.1 in special classes of dynamical systems. The system (X, T) is *totally minimal* if for all $n \in \mathbb{N}$, the system (X, T^n) is minimal. Our main result for totally minimal systems makes use of the *multiplicative upper Banach density*, defined for $A \subseteq \mathbb{N}$ by

$$(1) \quad d_{\times}^*(A) := \limsup_{n \rightarrow \infty} \max_{m \in \mathbb{N}} \frac{|A \cap \{mp_1^{e_1} \cdots p_n^{e_n} \mid e_1, \dots, e_n \in \{1, \dots, n\}\}|}{n^n},$$

where $(p_n)_{n \in \mathbb{N}}$ is an enumeration of the primes. This density, introduced and studied by Bergelson [Ber], is the multiplicative analogue of the additive upper Banach density in \mathbb{N} . It is independent of the chosen enumeration of the primes; see Definition 2.2 for an equivalent definition of d_{\times}^* and the remark following it.

Theorem 1.2. *Let (X, T) be a totally minimal dynamical system. There exists a residual set $X' \subseteq X$ such that for all non-empty, open $U \subseteq X$, there exists $\eta > 0$ such that for all $x \in X'$, the set $R(x, U)$ satisfies $d_{\times}^*(R(x, U)) \geq \eta$.*

Szemerédi's theorem [Sze] on arithmetic progressions can be used to prove that any set of positive multiplicative upper Banach density contains arbitrarily long geometric progressions; see Theorem 2.4. Therefore, Theorem 1.2 strengthens Theorem 1.1 when the system (X, T) is totally minimal. In fact, sets of positive multiplicative upper Banach density contain *geo-arithmetic configurations*, combinatorial configurations of the form $\{c(a + id)^j \mid 1 \leq i, j \leq \ell\}$ that are much richer than simply geometric progressions; see Theorem 2.5.

Without the assumption of total minimality, *local obstructions* appear that prevent return time sets from having positive multiplicative density. For instance, the set $4\mathbb{N} - 2$ is a set of return times in a four point rotation (which is minimal but not totally minimal), but it has zero multiplicative upper Banach density. The set $4\mathbb{N} - 2$ is, however, multiplicatively large in a different sense: it is a coset of the multiplicative subsemigroup $2\mathbb{N} - 1$.

We resolve local obstructions by measuring multiplicative density not along \mathbb{N} , but along cosets of multiplicative subsemigroups of \mathbb{N} . A *multiplicative subsemigroup* of \mathbb{N} is a subset $S \subseteq \mathbb{N}$ that is closed under multiplication, and a *coset* of S is a set of the form nS for $n \in \mathbb{N}$. The multiplicative upper Banach density d_{nS}^* for subsets of nS can be defined analogously to d_{\times}^* in (1) (using dilates of so-called *Følner sequences* in S) or as in Definition 2.2. For convenience, when $A \subseteq \mathbb{N}$, we write $d_{nS}^*(A)$ to mean $d_{nS}^*(A \cap nS)$.

For the special class of distal systems, we prove an analogue of Theorem 1.2 without the assumption of total minimality. A system (X, T) is called *distal* if for all $x, y \in X$ with $x \neq y$, $\inf_{n \in \mathbb{N}} d(T^n x, T^n y) > 0$. Distal systems encompass limits of group extensions of group rotations and form important building blocks in the various structure theories of minimal dynamical systems.

The following theorem shows that the local obstructions described above are the only types of obstructions to positive multiplicative density in distal systems.

Theorem 1.3. *Let (X, T) be a minimal distal system. There exists a residual set $X' \subseteq X$ such that for all non-empty, open $U \subseteq X$, there exists $\eta > 0$ such that for all $x \in X'$, there exists a multiplicative subsemigroup S of \mathbb{N} and $n \in \mathbb{N}$ such that the set $R(x, U)$ satisfies $d_{nS}^*(R(x, U)) \geq \eta$.*

Sets with positive multiplicative density along a coset of a multiplicative subsemigroup contain an abundance of multiplicative configurations, including arbitrarily

long geometric progressions and geo-arithmetic configurations; see Theorems 2.4 and 2.5.

Our final main result is purely combinatorial but indirectly concerns nilsystems, a subclass of distal systems that encompasses algebraic group extensions of group rotations.¹ A subset of \mathbb{N} is called IP_r , $r \in \mathbb{N}$, if it contains a set of the form

$$(2) \quad \left\{ \sum_{i \in I} x_i \mid \emptyset \neq I \subseteq \{1, \dots, r\} \right\}, \quad x_1, \dots, x_r \in \mathbb{N}.$$

A subset of \mathbb{N} is called IP_r^* if it has non-empty intersection with every IP_r set in \mathbb{N} . Such sets arose first in the work of Furstenberg and Katznelson [FK] on the multidimensional IP Szemerédi theorem and were recently used by Bergelson and Leibman [BL] to characterize nilsystems: roughly speaking, a system (X, T) is a nilsystem if and only if for all non-empty, open $U \subseteq X$, there exists $r \in \mathbb{N}$ such that for every $x \in U$, the set $R(x, U)$ is IP_r^* .

The following theorem addresses the multiplicative properties of additive translates of IP_r^* sets.

Theorem 1.4. *Let $A \subseteq \mathbb{N}$ be an IP_r^* set. For all $t \in \mathbb{Z}$, there exists a multiplicative subsemigroup S of \mathbb{N} and $n \in \mathbb{N}$ such that $d_{nS}^*(A + t) > 0$.*

Whereas the previous results rely on tools and techniques from dynamics, the statement and proof of Theorem 1.4 are entirely combinatorial. This imparts two advantages: we avoid the machinery necessary to work with nilsystems, and the result concerns a wider class of sets.² In particular, our result implies that return time sets in minimal nilsystems contain arbitrarily large geo-arithmetic configurations, hence arbitrarily long geometric progressions. When applied to sets of natural numbers that arise in polynomial Diophantine approximation, Theorem 1.4 yields the following corollary. Denote by $\{x\}$ the fractional part of $x \in \mathbb{R}$.

Corollary 1.5. *Let $p_1, \dots, p_k \in \mathbb{R}[x]$ be non-constant polynomials that are linearly independent in the following sense: for all $h_1, \dots, h_k \in \mathbb{Z}$, not all zero, at least one of the non-constant coefficients of $\sum_{i=1}^k h_i p_i$ is irrational. Let $I_1, \dots, I_k \subseteq [0, 1)$ be sets that are open when $[0, 1)$ is identified with the 1-torus. The set*

$$A := \{n \in \mathbb{Z} \mid \text{for all } i \in \{1, \dots, k\}, \{p_i(n)\} \in I_i\}$$

has positive multiplicative upper Banach density in a coset of a multiplicative subsemigroup of \mathbb{N} . As a consequence, for all $n \in \mathbb{N}$, there exist $a, c, d \in \mathbb{N}$ such that $\{c(a + id)^j \mid 1 \leq i, j \leq n\} \subseteq A$.

1.2. Motivation and historical context. Van der Waerden’s theorem on arithmetic progressions [vdW] is one of the most celebrated results in Ramsey Theory. An equivalent formulation due to Kakeya and Morimoto [KM, Theorem I] states that every syndetic subset of \mathbb{N} contains arbitrarily long arithmetic progressions. The multiplicative analogue of this result asserts that every *multiplicatively syndetic* subset of \mathbb{N} (that is, a set $A \subseteq \mathbb{N}$ for which there exists $N \in \mathbb{N}$ such that $A \cup A/2 \cup \dots \cup A/N = \mathbb{N}$, where $A/n := \{m \in \mathbb{N} \mid mn \in A\}$) contains arbitrarily long *geometric progressions*.

¹A *nilsystem* is a topological dynamical system (X, T) where X is a compact homogeneous space of a nilpotent Lie group G and T is a translation of X by an element of G .

²While every set of the form $R(x, U)$ in a minimal nilsystem is IP_r^* for some $r \in \mathbb{N}$, not every IP_r^* set in \mathbb{N} contains a set of return times from a minimal nilsystem; see Example 8.4.

The following long-standing open problem in Ramsey Theory features both of these additive and multiplicative notions and is the primary motivation for our work.

Question 1.6 ([BBHS]). *Does every additively syndetic subset of \mathbb{N} contain arbitrarily long geometric progressions?*

Just as in other problems in Ramsey theory involving both addition and multiplication – most notably the $\{x+y, xy\}$ problem that was recently resolved in [Mor] – analysis is complicated by the combination of addition and multiplication. Until now, very little progress has been made on Question 1.6; in fact, it is still unknown whether or not syndetic subsets of \mathbb{N} contain a square integer ratio. Recent work in [Pat] addresses the set of integer ratios of elements of syndetic sets.

Being unable to make progress on the problem in its full generality, it is natural to restrict the class of syndetic subsets under consideration. Each of our main results concerns such a restriction: Theorem 1.1 lends some evidence toward a positive answer to Question 1.6 by showing that many syndetic sets of dynamical origin contain arbitrarily long geometric progressions; Theorems 1.2 and 1.3 show that much more is true with further restrictions on the dynamics: syndetic sets arising from these systems have positive multiplicative density; and Theorem 1.4 shows that members of a combinatorially defined subclass of syndetic sets also have positive multiplicative density.

The idea to approach problems in Ramsey Theory and combinatorial number theory with tools from dynamics goes back to the work of Furstenberg [Fur2] in the measure-theoretic setting and Furstenberg and Weiss [FW] in the topological setting. The basic idea is that the existence of combinatorial configurations in subsets of \mathbb{N} can be reformulated in the language of dynamics to be about the recurrence of points and sets. Consider, for example, that the set $A \subseteq \mathbb{N}$ contains an arithmetic progression of length $k+1$ and step size n if and only if

$$(3) \quad A \cap (A - n) \cap (A - 2n) \cap \cdots \cap (A - kn) \neq \emptyset.$$

Results concerning the recurrence of sets in topological dynamical systems can be made to apply to sets in the positive integers via *correspondence principles* which, roughly speaking, turn the would-be dynamical system $(\mathbb{N}, n \mapsto n+1)$ into a genuine one and convert the expression (3) into one similar to (4) below regarding the recurrence of open sets.

Exemplifying this approach, the following topological dynamical result implies (and, in fact, can be shown to be equivalent to) van der Waerden's theorem.

Theorem 1.7 ([FW, Theorem 1.5]). *Let (X, T) be a minimal dynamical system. For all non-empty, open $U \subseteq X$ and for all $k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that*

$$(4) \quad U \cap T^{-n}U \cap \cdots \cap T^{-kn}U \neq \emptyset.$$

Interestingly, Theorem 1.7 can be reformulated in terms of return time sets and multiplicative density. A set $A \subseteq \mathbb{N}$ has multiplicative upper Banach density equal to 1, $d_{\times}^*(A) = 1$, if and only if for all finite $F \subseteq \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $nF \subseteq A$. (This equivalence is not apparent from (1) but is immediate from Definition 2.2. Such sets are called *multiplicatively thick*.) We demonstrate the equivalence between Theorem 1.7 and the following theorem at the beginning of Section 5.

Theorem 1.8. *Let (X, T) be a minimal dynamical system. There exists a residual set $X' \subseteq X$ such that for all $x \in X'$ and all non-empty, open $U \subseteq X$ containing x , the set $R(x, U)$ satisfies $d_x^*(R(x, U)) = 1$, i.e., it is multiplicatively thick.*

Though Theorem 1.8 strongly resembles Theorem 1.1 – the set U is not required to be a neighborhood of the point x in Theorem 1.1 – one cannot hope to easily derive the latter from the former by translating the return time sets. Evidence for this is given by the fact that there are examples of sets which are multiplicatively large but whose additive translates are all multiplicatively very small; one can construct, for example, a multiplicatively thick set $A \subseteq \mathbb{N}$ with the property that for all $t \in \mathbb{Z} \setminus \{0\}$, the set $A + t$ has zero multiplicative density in all cosets of all non-trivial multiplicative subsemigroups of \mathbb{N} .

The following theorem, a result of Glasner's reformulated in a similar vein, is an improvement to Theorem 1.7 in the case that (X, T) is weakly mixing. A system (X, T) is *weakly mixing* if the system $(X^2, T \times T)$ contains a point with a dense forward orbit. A minimal, weakly mixing system is totally minimal, so our Theorem 1.2 can be understood to make less of an assumption on the dynamics and arrive at a similar, but weaker, conclusion.

Theorem 1.9 ([Gla1, Corollary 2.5]). *Let (X, T) be a minimal, weakly mixing dynamical system. There exists a residual set $X' \subseteq X$ such that for all $x \in X'$ and all non-empty, open sets $U \subseteq X$, the set $R(x, U)$ satisfies $d_x^*(R(x, U)) = 1$.*

These theorems exemplify the historical precedent that motivated our approach to Question 1.6 by considering syndetic sets arising in dynamics. As Question 1.6 is purely combinatorial, our ultimate goal is to understand the multiplicative configurations contained in arbitrary syndetic subsets of the natural numbers. If one is looking for additive configurations in syndetic sets then this is achieved historically via dynamics by making use of translation invariance: since arithmetic progressions and additive density are translation invariant, one can transfer the problem of finding such configurations (as in van der Waerden's theorem) to an analogous dynamical problem on symbolic shift space. The same approach does not work as easily for multiplicative configurations. The fact that geometric progressions and multiplicative density are not translation invariant presents the most serious obstacle faced in this work.

1.3. Outline of the paper. In Section 2, we gather definitions and prove some initial lemmas on additive and multiplicative density, topological and measurable dynamics, and set-valued maps. We define the rational topological Kronecker factor of a system in Section 3 and prove a key lemma about distal systems. In Section 4, we establish some preliminary results concerning dynamics on the orbit closure of the diagonal. This is followed by proofs of the main results, Theorem 1.1, Theorems 1.2 and 1.3, and Theorem 1.4, in Sections 5, 6, and 7, respectively. We conclude the work with Sections 8 and 9 by exhibiting some syndetic sets which do not arise from dynamics and collecting some questions for further consideration.

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2. DEFINITIONS AND PRELIMINARY RESULTS

In this section we gather definitions and preliminary results that will be necessary later on. Denote by \mathbb{N} the set of positive integers and by \mathbb{N}_0 the set $\mathbb{N} \cup \{0\}$.

2.1. Set algebra, additive and multiplicative density. For $A \subseteq \mathbb{N}$ and $n \in \mathbb{N}$, define

$$A - n := \{m \in \mathbb{N} \mid m + n \in A\} \text{ and } A/n := \{m \in \mathbb{N} \mid mn \in A\}.$$

The set A is *syndetic* if there exists a finite set $F \subseteq \mathbb{N}$ for which

$$A - F := \bigcup_{n \in F} (A - n) = \mathbb{N}.$$

This is equivalent to the set A having bounded gaps: if $A = \{a_1 < a_2 < \dots\}$, then A is syndetic if and only if $\sup_{i \in \mathbb{N}} (a_{i+1} - a_i)$ is finite.

A *mean* on \mathbb{N} is a positive linear functional of norm 1 on $B(\mathbb{N})$, the Banach space of bounded, real-valued functions on \mathbb{N} with the supremum norm. A mean λ is (*additively*) *translation invariant* if for all $f \in B(\mathbb{N})$ and all $m \in \mathbb{N}$, $\lambda(n \mapsto f(n + m)) = \lambda(f)$. Abusing notation, for $A \subseteq \mathbb{N}$, we write $\lambda(A)$ to mean $\lambda(\mathbb{1}_A)$, where $\mathbb{1}_A \in B(\mathbb{N})$ is the indicator function of A ; if λ is translation invariant, then for all $m \in \mathbb{N}$, $\lambda(A - m) = \lambda(A + m) = \lambda(A)$.

The following is an easy consequence of the pigeonhole principle that will be used repeatedly throughout this work.

Lemma 2.1. *Let λ be a mean on \mathbb{N} and $A_1, \dots, A_k \subseteq \mathbb{N}$. If $\eta > 0$ is such that for all $i \in \{1, \dots, k\}$, $\lambda(A_i) > \eta$, then there exists $n \in \mathbb{N}$ for which $|\{1 \leq i \leq k \mid n \in A_i\}| > \eta k$.*

Proof. Define $g(n) = \sum_{i=1}^k \mathbb{1}_{A_i}(n)/k$. Because λ is positive and $\lambda(g) > \eta$, there exists $n \in \mathbb{N}$ for which $g(n) > \eta$, as was to be shown. \square

A *multiplicative subsemigroup* S of \mathbb{N} is a subset of \mathbb{N} that is closed under multiplication, and a *coset* of S is a set of the form nS for $n \in \mathbb{N}$. The multiplicative subsemigroups that will appear most frequently in this paper are

$$S_N := \{n \in \mathbb{N} \mid (n, N) = 1\},$$

the natural numbers coprime to a given positive integer $N \in \mathbb{N}$.

Definition 2.2. Let S be a multiplicative subsemigroup of \mathbb{N} , $n \in \mathbb{N}$, and $A \subseteq nS$. The *multiplicative upper Banach density* of A in nS is

$$d_{nS}^*(A) = \sup \{ \alpha \geq 0 \mid \forall F \subseteq nS \text{ finite, } \exists s \in S, |sF \cap A| \geq \alpha |F| \}.$$

When $S = \mathbb{N}$, we write $d_{\mathbb{N}}^*$ instead of $d_{\mathbb{N}}^*$. When $A \subseteq \mathbb{N}$ is not a subset of nS , the symbol $d_{nS}^*(A)$ is understood to mean $d_{nS}^*(A \cap nS)$. Note that for all $A \subseteq \mathbb{N}$, $d_{nS}^*(A) = d_S^*(A/n)$.

Remark 2.3. The upper Banach density d_S^* that was just defined for multiplicative subsemigroups S is equivalent to the upper Banach density defined via multiplicative Følner sequences for (S, \cdot) (as in (1) in the case of (\mathbb{N}, \cdot)) or multiplicatively invariant means on S . For a proof, see [BG, Theorem 3.5].

A set $A \subseteq \mathbb{N}$ is *GP-rich* if it contains arbitrarily long geometric progressions, subsets of the form $\{nm, nm^2, \dots, nm^\ell\}$. The following theorem is a simple consequence of Definition 2.2 and Szemerédi's theorem [Sze] on arithmetic progressions.

Theorem 2.4. *Let S be a multiplicative subsemigroup of \mathbb{N} , $n \in \mathbb{N}$, and $A \subseteq nS$. If $d_{nS}^*(A) > 0$, then A is GP-rich.*

Proof. Let $0 < \varepsilon < d_{nS}^*(A)$ and $\ell \in \mathbb{N}$. By Szemerédi's theorem, there exists $L \in \mathbb{N}$ so that all subsets of $\{1, \dots, L\}$ with relative density at least ε contain an arithmetic progression of length ℓ . It follows that for all $m, s \in \mathbb{N}$, all subsets of $s\{m, m^2, \dots, m^L\}$ of relative density at least ε contain a geometric progression of length ℓ .

Fix $m \in S$, and put $F = \{nm, nm^2, \dots, nm^L\} \subseteq nS$. By the definition of multiplicative upper Banach density, because $d_{nS}^*(A) > \varepsilon$, there exists $s \in S$ such that the set A has relative density at least ε in $\{snm, snm^2, \dots, snm^L\}$; in particular, A contains a geometric progression of length ℓ . \square

In fact, sets with positive multiplicative density contain much richer combinatorial configurations than simply geometric progressions. Bergelson [Ber] showed that a set $A \subseteq \mathbb{N}$ with positive multiplicative density contains *geo-arithmetic configurations* such as the ones appearing in the following theorem. We will use the following extension of his result for applications in this paper.

Theorem 2.5. *Let $n, N \in \mathbb{N}$, and $A \subseteq nS_N$. If $d_{nS_N}^*(A) > 0$, then for all $\ell \in \mathbb{N}$, there exist $a, c, d \in \mathbb{N}$ such that $\{c(a + id)^j \mid 1 \leq i, j \leq \ell\} \subseteq A$.*

Proof. It is quick to verify that the multiplicative subsemigroup S_N satisfies the conditions in [BG, Theorem 8.8]. Let $\ell \in \mathbb{N}$, and apply that theorem to A/n (using that $d_{S_N}^*(A/n) > 0$) with the endomorphisms $\varphi_i : S_N \rightarrow S_N$ defined by $\varphi_i(m) = m^i$ and with the finite set F equal to an arithmetic progression long enough to guarantee that the subset F' (in the notation of [BG, Theorem 8.8]) contains an arithmetic progression of length ℓ . This yields the desired configuration in the set A/n ; multiplying by n yields the configuration in the set A . \square

2.2. Topological and measurable dynamics. Let X and Y be topological spaces and $A \subseteq X$. The set A is *residual* if it contains a dense G_δ set, and it is *meager* if its complement is residual. A map $f : X \rightarrow Y$ is *semiopen* if $f(A)$ has non-empty interior when A has non-empty interior.

Lemma 2.6. *Let X be a complete metric space and Y be a Hausdorff topological space. If $A \subseteq X$ is residual and $f : X \rightarrow Y$ is a continuous, semiopen surjection, then $f(A) \subseteq Y$ is residual.*

Proof. This is proved in [Phe, Lemma 4.25] under the assumption that f is an open map. The same proof, with the obvious adjustments (i.e., in the Banach-Mazur game, the winning strategy of B comes by choosing the open set V_i to be the set of interior points of $f(B_i)$, which is non-empty because f is semiopen) gives the result in the case that f is semiopen. \square

In a metric space (X, d) , the open ball of radius r centered at x will be denoted $B(x, r)$. The set A is ε -dense if for all $x \in X$, there exists $a \in A$ such that $d(x, a) < \varepsilon$.

A *topological dynamical system* (X, T) is a compact metric space X paired with a continuous map $T : X \rightarrow X$; we will usually refer to (X, T) as simply a *system*. The system (X, T) is *invertible* if T is a homeomorphism. The set A is *T invariant* if $TA \subseteq A$.

Given $x \in X$ and $U \subseteq X$, the *set of return times of x to U* is

$$R_T(x, U) = \{n \in \mathbb{N} \mid T^n x \in U\}.$$

The letter U will usually be used for a non-empty, open subset of X , and we will usually write $R(x, U)$ instead of $R_T(x, U)$ when the map T is understood. Note the standard manipulations

$$R_{T^n}(x, U) = R_T(x, U)/n \quad \text{and} \quad R_T(T^n x, U) = R_T(x, U) - n.$$

The (*forward*) *orbit of a point $x \in X$ under T* is $o_T(x) := \{T^n x \mid n \in \mathbb{N}_0\}$ while that of a subset $Y \subseteq X$ is $o_T(Y) := \bigcup_{n \in \mathbb{N}_0} T^n Y$. We denote the corresponding closures with $\bar{o}_T(x)$ and $\bar{o}_T(Y)$ respectively. A system (X, T) is *minimal* if all points have a dense forward orbit and *totally minimal* if for all $n \in \mathbb{N}$, the system (X, T^n) is minimal.

Given continuous maps $T, S : X \rightarrow X$ that commute, the (*forward*) *orbit of a point $x \in X$ under T and S* is $o_{T,S}(x) = \{T^n S^m x \mid n, m \in \mathbb{N}_0\}$, and we denote its closure with $\bar{o}_{T,S}(x)$. The system (X, T, S) is *minimal* if all points have a dense forward orbit. In the case that T and S are invertible, minimality is equivalent by the following lemma to all points having a dense “two-sided” orbit. We will make use of this lemma in Section 4.

Lemma 2.7. *Let X be a compact metric space, and let S be a commutative sub-semigroup of the group of homeomorphisms of X . Denote by G the group generated by S . The following are equivalent:*

- (1) *For all $x \in X$, $\overline{Sx} = X$.*
- (2) *For all $x \in X$, $\overline{Gx} = X$.*

Proof. Clearly (1) implies (2) since $S \subseteq G$. Suppose (2) holds, and let $x \in X$. By adjoining the identity map to S if necessary, we may assume without loss of generality that S contains the identity. Let

$$A = \bigcap_{s \in S} \overline{Ssx}.$$

We will show that $A = X$; this will conclude the proof since $A \subseteq \overline{Sx}$. Since the collection $\{\overline{Ssx} \mid s \in S\}$ has the finite intersection property and X is compact, A is compact and non-empty. We claim that for all $g \in G$, $gA \subseteq A$. Let $g \in G$; since S generates G and is commutative, there exist $s_1, s_2 \in S$ such that $g = s_1 s_2^{-1}$. Since $g : X \rightarrow X$ is a homeomorphism,

$$gA = \bigcap_{s \in S} g \overline{Ssx} = \bigcap_{s \in S} \overline{Ssgx}.$$

Let $y \in gA$; we will show that $y \in A$. Let $s \in S$. Since $y \in gA$ and $s_2 s \in S$, $y \in \overline{Ss_2 s s_1 s_2^{-1} x}$. Since $Ss_1 \subseteq S$, we see that $y \in \overline{Ssx}$, and since $s \in S$ was arbitrary, this shows $y \in A$. Since $y \in gA$ was arbitrary, $gA \subseteq A$. Now by (2), for any $a \in A$, $X = \overline{Ga} \subseteq A$, meaning $A = X$, as was to be shown. \square

The system (X, T) is *distal* if for all $x, y \in X$ with $x \neq y$, $\inf_{n \in \mathbb{N}} d(T^n x, T^n y) > 0$. Distal systems appear in the statement of Theorem 1.3. Distality is a fundamental concept in understanding the structure of topological and measurable dynamical

systems; see [Gla2] and the references therein. In this work, the definition of distality and the fact that distal systems are invertible³ will suffice.

Definition 2.8 (cf. [Dow, Definition 6.8.10]). Let (X, T) be a system with T surjective. The *(topological) natural extension* of (X, T) is the system (W, T) , where

$$W := \{(w_i)_{i \in \mathbb{Z}} \in X^{\mathbb{Z}} \mid \forall i \in \mathbb{Z}, w_{i+1} = Tw_i\}$$

inherits the topology from the product topology on $X^{\mathbb{Z}}$, $T = \sigma|_W$ is the restriction of the left shift on $X^{\mathbb{Z}}$ to W , and $\pi : (W, T) \rightarrow (X, T)$ is defined by $\pi((w_i)_{i \in \mathbb{Z}}) = w_0$.

Lemma 2.9. *Let (X, T) be a system with T surjective, and let (W, T) be its natural extension.*

- (1) *The system (W, T) is invertible.*
- (2) *For all $n \in \mathbb{N}$, the system (X, T^n) is minimal if and only if (W, T^n) is minimal.*
- (3) *If (X, T) is minimal, then the factor map $(W, T) \rightarrow (X, T)$ is semiopen.*

Proof. Statement (1) follows from the definition of (W, T) .

Statement (2) follows from two facts: factors of minimal systems are minimal, and, when (X, T^n) is minimal, W is the only closed, T^n -invariant subset of W that surjects onto X .

Statement (3) follows from (2) and the more general fact that a factor map $\pi : (W, T) \rightarrow (X, T)$ of minimal systems is semiopen. Let $U \subseteq W$ be open, and let $V \subseteq U$ be closed with non-empty interior. Since the system (W, T) is minimal, there exists $h \in \mathbb{N}$ so that $\bigcup_{n=1}^h T^{-n}V = W$. Applying the factor map, $\bigcup_{n=1}^h T^{-n}\pi(V) = X$. Each $T^{-n}\pi(V)$ is closed, so by the Baire Category Theorem, there exists $n \in \{1, \dots, h\}$ so that $T^{-n}\pi(V)$ has non-empty interior. By [KST, Theorem 2.4], the map T is semiopen, hence T^n is semiopen, too. It follows that $T^n T^{-n}\pi(V) \subseteq \pi(V) \subseteq \pi(U)$ has non-empty interior, as was to be shown. \square

We will frequently make use of probability measures on compact metric spaces. Unless otherwise stated, all measures appearing in this work are Borel probability measures. If μ is a measure on X , we write $T\mu$ for the push-forward measure defined for $A \subseteq X$ by $T\mu(A) = \mu(T^{-1}A)$. The measure μ is *T -invariant* if $T\mu = \mu$.

We thank Joel Moreira for permission to include Lemmas 2.10 and 2.11, previously unpublished, in this paper.

Lemma 2.10. *Let (X, T) be a system and μ be a T -invariant probability measure on X . Suppose that $x \in X$ is such that $\bar{o}(x) = X$. For all $f \in C(X)$, $\varepsilon > 0$, and $N_0 \in \mathbb{N}$, there exist $m, N \in \mathbb{N}$ with $N \geq N_0$ such that*

$$\left| \int_X f d\mu - \frac{1}{N} \sum_{i=m}^{m+N-1} f(T^i x) \right| < \varepsilon.$$

Proof. Let $f \in C(X)$, $\varepsilon > 0$, and $N_0 \in \mathbb{N}$. Let $M = \max_{x \in X} |f(x)|$, $I = \int_X f d\mu$, and for $m, N \in \mathbb{N}$, put

$$A(m, N) := \frac{1}{N} \sum_{i=m}^{m+N-1} f(T^i x).$$

³This follows immediately from the fact that the Ellis enveloping semigroup is a group; see [Fur1, Theorem 3.1].

Note that for all $N > 2M/\varepsilon$,

$$(5) \quad \max \{|A(m, N) - A(m, N+1)|, |A(m, N) - A(m+1, N)|\} \leq \frac{2M}{N} < \varepsilon.$$

By the ergodic decomposition, there exist ergodic, T -invariant, Borel probability measures μ_1 and μ_2 such that

$$I_1 := \int_X f d\mu_1 \leq I \leq I_2 := \int_X f d\mu_2.$$

By [Fur3, Proposition 3.9], for each $i = 1, 2$, there exist $N_i > \max\{N_0, 2M/\varepsilon\}$ and $m_i \in \mathbb{N}$ such that $|I_i - A(m_i, N_i)| < \varepsilon$. It follows that

$$I - \varepsilon \leq I_2 - \varepsilon < A(m_2, N_2) \quad \text{and} \quad A(m_1, N_1) < I_1 + \varepsilon \leq I + \varepsilon.$$

By (5), there exist $m, N \in \mathbb{N}$ between m_1, m_2 and N_1, N_2 , respectively, for which $I - \varepsilon < A(m, N) < I + \varepsilon$, as was to be shown. \square

Lemma 2.11. *Let (X, T) be a system and μ be a T -invariant probability measure on X . Suppose that $x \in X$ is such that $\bar{o}(x) = X$. There exists an additively invariant mean λ on \mathbb{N} such that for all $f \in C(X)$,*

$$\lambda(n \mapsto f(T^n x)) = \int_X f d\mu.$$

Proof. Let

$$\begin{aligned} \Omega &:= \{\omega_f : n \mapsto f(T^n x) \mid f \in C(X)\} \subseteq B(\mathbb{N}), \\ \chi &:= \{n \mapsto \zeta(n+1) - \zeta(n) \mid \zeta \in B(\mathbb{N})\} \subseteq B(\mathbb{N}). \end{aligned}$$

Note that because $\bar{o}(x) = X$, the map $f \mapsto \omega_f$ is a bijection from $C(X)$ to Ω . In what follows, when we write $\omega_f \in \Omega$, we are implicitly specifying both $\omega_f \in \Omega$ and the corresponding $f \in C(X)$.

We claim that for all $\omega_f \in \Omega \cap \chi$, $\int_X f d\mu = 0$. Indeed, there exists $\zeta \in B(\mathbb{N})$ such that for all $n \in \mathbb{N}$,

$$w_f(n) = f(T^n x) = \zeta(n+1) - \zeta(n).$$

Because ζ is bounded, for all $\varepsilon > 0$, all sufficiently large $N \in \mathbb{N}$, and all $m \in \mathbb{N}$,

$$\left| \frac{1}{N} \sum_{n=m}^{m+N-1} f(T^n x) \right| = \frac{|\zeta(m+N) - \zeta(m)|}{N} < \varepsilon.$$

It follows now by Lemma 2.10 that $\int_X f d\mu = 0$.

Define a linear function $\lambda : \Omega + \chi \rightarrow \mathbb{R}$ by $\lambda(\omega_f + c) = \int_X f d\mu$. The previous paragraph shows that λ is well defined. The plan is to extend λ using the Hahn-Banach theorem to a positive linear functional; any such extension of λ will satisfy the conclusions of the lemma.

First, we claim that $\|\lambda\| \leq 1$. To see this, let $\tau = \omega_f + c \in \Omega + \chi$, and let $\zeta \in B(\mathbb{N})$ be such that $c(n) = \zeta(n+1) - \zeta(n)$. By Lemma 2.10, for all $\varepsilon > 0$, there exist $N > \|\zeta\|/\varepsilon$ and $m \in \mathbb{N}$ such that

$$\left| \int_X f d\mu - \frac{1}{N} \sum_{n=m}^{m+N-1} \omega_f(n) \right| < \varepsilon.$$

By the same reasoning as before,

$$\left| \frac{1}{N} \sum_{n=m}^{m+N-1} (\tau(n) - \omega_f(n)) \right| = \left| \frac{1}{N} \sum_{n=m}^{m+N-1} c(n) \right| = \frac{|\zeta(m+N) - \zeta(m)|}{N} < 2\varepsilon.$$

It follows that

$$\left| \lambda(\tau) - \frac{1}{N} \sum_{n=m}^{m+N-1} \tau(n) \right| = \left| \int_X f \, d\mu - \frac{1}{N} \sum_{n=m}^{m+N-1} \tau(n) \right| < 3\varepsilon.$$

This shows that there exists $n \in \{m, \dots, m+N-1\}$ for which $|\tau(n)| \geq |\lambda(\tau)| - 3\varepsilon$, meaning $\|\tau\| \geq |\lambda(\tau)| - 3\varepsilon$. Since $\varepsilon > 0$ was arbitrary, $|\lambda(\tau)| \leq \|\tau\|$.

By the Hahn-Banach theorem, λ extends to a linear functional on $B(\mathbb{N})$ (which we still call λ) with norm $\|\lambda\| \leq 1$. We have only to show that λ is positive and translation invariant. To show positivity, let $\mathbb{1} \in B(\mathbb{N})$ denote the constant one function, and note that $\tau(\mathbb{1}) = 1$. Suppose $\tau \in B(\mathbb{N})$ is positive. Since $0 \leq \tau/\|\tau\| \leq 1$, we have $|\lambda(\mathbb{1} - \tau/\|\tau\|)| \leq 1$, and it follows that

$$\frac{\lambda(\tau)}{\|\tau\|} = \lambda\left(\frac{\tau}{\|\tau\|}\right) = 1 - \lambda\left(\mathbb{1} - \frac{\tau}{\|\tau\|}\right) \geq 0.$$

To show invariance, let $\tau \in B(\mathbb{N})$, and define $c \in B(\mathbb{N})$ by $c(n) = \tau(n+1) - \tau(n)$. Since $c \in \chi$, $\lambda(c) = 0$, meaning $\lambda(n \mapsto \tau(n+1)) = \lambda(\tau)$. \square

2.3. Set-valued maps. Let (X, d) be a compact metric space. For $A \subseteq X$ and $\delta > 0$, let

$$(6) \quad [A]_\delta := \{x \in X \mid \exists a \in A, d(x, a) \leq \delta\}.$$

The set of all non-empty, closed subsets of X is denoted by $\mathcal{F}(X)$. The Hausdorff metric, defined between $F, H \in \mathcal{F}(X)$ by

$$d_H(F, H) := \inf\{\delta > 0 \mid F \subseteq [H]_\delta \text{ and } H \subseteq [F]_\delta\},$$

makes $(\mathcal{F}(X), d_H)$ a compact metric space.

Definition 2.12. Let X and Y be compact metric spaces. A map $\varphi : X \rightarrow \mathcal{F}(Y)$ is *lower semicontinuous* (LSC) at $x \in X$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for all $x' \in X$ with $d(x, x') < \delta$, $\varphi(x) \subseteq [\varphi(x')]_\varepsilon$.

Lemma 2.13. *Let (X, T) be a system. The map $\bar{o}_T : X \rightarrow \mathcal{F}(X)$ is LSC. In particular, it is Borel measurable: for all Borel subsets $B \subseteq \mathcal{F}(X)$, the set $\bar{o}_T^{-1}(B) \subseteq X$ is Borel.*

Proof. For convenience, we will write \bar{o} in place of \bar{o}_T . Let $x \in X$ and $\varepsilon > 0$. Since $\bar{o}(x)$ is compact, there exist $m_1, \dots, m_k \in \mathbb{N}_0$ such that $\{T^{m_1}x, \dots, T^{m_k}x\}$ is an $\varepsilon/2$ -dense subset of $\bar{o}(x)$. Because each T^{m_i} is continuous, there exists $\delta > 0$ such that for all $x' \in X$ with $d(x, x') < \delta$, for all $1 \leq i \leq k$, $d(T^{m_i}x, T^{m_i}x') < \varepsilon/2$.

We claim now that for all $x' \in B(x, \delta)$, $\bar{o}(x) \subseteq [\bar{o}(x')]_\varepsilon$. Let $x' \in B(x, \delta)$ and $y \in \bar{o}(x)$. There exists $1 \leq i \leq k$ for which $d(T^{m_i}x, y) < \varepsilon/2$, and by the triangle inequality, $d(T^{m_i}x', y) < \varepsilon$. This means $y \in [\bar{o}(x')]_\varepsilon$, as was to be shown.

The second statement follows from the fact that when X and Y are compact metric spaces, all LSC functions $\varphi : X \rightarrow \mathcal{F}(Y)$ are Borel measurable; see Lemma 17.5, Theorem 17.15, and Theorem 18.10 in [AB]. \square

Lemma 2.14. *Let (X, T) be an invertible system, and denote by Ξ the set of points of discontinuity of the map $\bar{\sigma}_T : X \rightarrow \mathcal{F}(X)$. There exists a countable family $\{B_i\}_{i \in \mathbb{N}}$ of closed, T -invariant, empty-interior subsets of X for which $\Xi \subseteq \bigcup_i B_i$.*

Proof. For convenience, we will write $\bar{\sigma}$ in place of $\bar{\sigma}_T$. For $A \in \mathcal{F}(X)$ and $\varepsilon > 0$, let $M(A, \varepsilon)$ be the largest positive integer n for which there exist $a_1, \dots, a_n \in A$ satisfying, for all $i \neq j$, $d(a_i, a_j) > \varepsilon$, and let $U(A, \varepsilon) = \{x \in X \mid \exists a \in A, d(x, a) < \delta\}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, let

$$B_{n,\varepsilon} := \left\{ x \in X \mid \begin{array}{l} M(\bar{\sigma}(x), \varepsilon) \leq n, \text{ and} \\ \forall \varepsilon' \in (0, 3\varepsilon), \forall U \ni x \text{ open}, \exists y \in U, \bar{\sigma}(y) \not\subseteq U(\bar{\sigma}(x), \varepsilon') \end{array} \right\}.$$

It is proved in [For, Theorem 1] that each $B_{n,\varepsilon}$ is closed with empty interior and that $\Xi \subseteq \bigcup\{B_{n,\varepsilon} \mid n \in \mathbb{N}, \varepsilon \in \mathbb{Q}_+\}$.

It remains to be shown that each $B_{n,\varepsilon}$ is T -invariant, i.e., $TB_{n,\varepsilon} \subseteq B_{n,\varepsilon}$. Let $x \in B_{n,\varepsilon}$. Since $\bar{\sigma}(Tx) \subseteq \bar{\sigma}(x)$, $M(\bar{\sigma}(Tx), \varepsilon) \leq M(\bar{\sigma}(x), \varepsilon) \leq n$. Let $0 < \varepsilon' < 3\varepsilon$ and W be an open neighborhood of Tx . Since $x \in T^{-1}W$ and $x \in B_{n,\varepsilon}$, there exists $y \in T^{-1}W$ with $d(x, y) < \varepsilon'$ such that $\bar{\sigma}(y) \not\subseteq U(\bar{\sigma}(x), \varepsilon')$. Consider $Ty \in W$; it will complete the proof to show that $\bar{\sigma}(Ty) \not\subseteq U(\bar{\sigma}(Tx), \varepsilon')$. Since $d(x, y) < \varepsilon'$, $y \in U(\bar{\sigma}(x), \varepsilon')$. Since $\bar{\sigma}(y) \not\subseteq U(\bar{\sigma}(x), \varepsilon')$, it follows that $\bar{\sigma}(Ty) \not\subseteq U(\bar{\sigma}(x), \varepsilon')$. Because $U(\bar{\sigma}(Tx), \varepsilon') \subseteq U(\bar{\sigma}(x), \varepsilon')$, this implies that $\bar{\sigma}(Ty) \not\subseteq U(\bar{\sigma}(Tx), \varepsilon')$. \square

Lemma 2.15. *Let X be a compact metric space. Suppose $(\mu_n)_{n \in \mathbb{N}}$ is a sequence of Borel probability measures converging in the weak-* topology to a probability measure μ . If $(H_n)_{n \in \mathbb{N}}$ is a sequence of closed subsets of X such that $\text{supp } \mu_n \subseteq H_n$ and H is a closed subset of X such that $H_n \rightarrow H$ in the Hausdorff metric, then $\text{supp } \mu \subseteq H$.*

Proof. We must prove that $\mu(H) = 1$. Since $H = \bigcap_{n \in \mathbb{N}} [H]_{1/n}$, in order to prove that $\mu(H) = 1$, it suffices to prove that for all $\delta > 0$, $\mu([H]_\delta) = 1$.

Fix $\delta > 0$. Convergence in the Hausdorff metric implies that $H_n \subseteq [H]_\delta$ for all sufficiently large $n \in \mathbb{N}$. By the properties of weak convergence of measures,

$$\mu([H]_\delta) \geq \limsup_{n \rightarrow \infty} \mu_n([H]_\delta) \geq \limsup_{n \rightarrow \infty} \mu_n(H_n) = 1,$$

meaning $\mu([H]_\delta) = 1$, as was to be shown. \square

3. THE RATIONAL TOPOLOGICAL KRONECKER FACTOR

Let (X, T) be a minimal system. According to [Ye, Theorem 3.1], for all $n \in \mathbb{N}$, the set X decomposes into a disjoint union of $d_n = d_n(X, T) \in \mathbb{N}$ clopen sets

$$X = X_{n,0} \cup \dots \cup X_{n,d_n-1},$$

where d_n divides n , for all $k \in \mathbb{Z}$, $T^k X_{n,j} = X_{n,j+k \pmod{d_n}}$, and the systems $(X_{n,j}, T^n)$ are minimal. To save on notation, the second component of the index on $X_{n,j+k}$ and on related expressions will be implicitly understood to be taken modulo d_n . The notation $(X_{N,j})_{n,i}$ will mean the i^{th} of the $d_n(X_{N,j}, T^N)$ many T^{Nn} -minimal components of the system $(X_{N,j}, T^N)$. With this definition, it is quick to check that $(X_{N,j})_{n,i} = X_{nN,iN+j}$.

For $U \subseteq X$ and a probability measure μ on X , we write

$$(2) \quad U_{n,i} := U \cap X_{n,i}, \text{ and } \mu_{n,i} := d_n \mu|_{X_{n,i}}.$$

Note that if μ is T -invariant, then $T\mu_{n,i} = \mu_{n,i+1}$ and $\mu_{n,i}$ is T^n -invariant. Though $\mu_{n,i}$ is technically a measure on X , we will sometimes regard $\mu_{n,i}$ as a measure

on $X_{n,i}$ so that $(X_{n,i}, T^n, \mu_{n,i})$ is a measure preserving system. This allows us to define the symbol $(\mu_{N,j})_{n,i}$ as in (7); regarded as measures on X , it is quick to check that $(\mu_{N,j})_{n,i} = \mu_{nN, iN+j}$.

Lemma 3.1. *Let (X, T) be a minimal system and $U \subseteq X$ be a non-empty, open set. There exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $(n, N) = 1$ and all $x \in X$, the T^n -orbit closure of x has non-empty intersection with U .*

Proof. It is equivalent to show that there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $(n, N) = 1$, $\bigcup_{j=1}^{\infty} (T^n)^{-j}U = X$.

Because T is minimal, there exists $h \in \mathbb{N}$ for which $X = \bigcup_{j=1}^h T^{-j}U$. The conclusion of the lemma will follow if we show that the set

$$B := \left\{ n \in \mathbb{N} \mid \bigcup_{j=1}^{\infty} (T^n)^{-j}U \neq X \right\}$$

does not contain h pairwise coprime elements.

Let $n \in B$. There exists $0 \leq i_n \leq d_n - 1$ such that $U_{n, i_n} = \emptyset$. Indeed, if this was not the case, then for all $0 \leq i \leq d_n - 1$, the set $U_{n, i}$ would be a non-empty, open subset of $X_{n, i}$. It would follow by the minimality of $(X_{n, i}, T^n)$ that

$$X = \bigcup_{i=0}^{d_n-1} X_{n, i} = \bigcup_{i=0}^{d_n-1} \bigcup_{j=1}^{\infty} (T^n)^{-j}U_{n, i} = \bigcup_{j=1}^{\infty} (T^n)^{-j}U,$$

contradicting the fact that $n \in B$.

Suppose for a contradiction that $n_1, \dots, n_h \in B$ are pairwise coprime. Since d_{n_i} divides n_i , the numbers d_{n_1}, \dots, d_{n_h} are also pairwise coprime. For each $1 \leq j \leq h$, let $0 \leq i_{n_j} \leq d_{n_j} - 1$ be the index for which $U_{n_j, i_{n_j}} = \emptyset$. By the Chinese Remainder Theorem, there exists $\ell \in \mathbb{N}$ such that for all $1 \leq j \leq h$, $\ell + j \equiv i_{n_j} \pmod{d_{n_j}}$. Using the fact that $U_{n_j, i_{n_j}} = \emptyset$,

$$(T^{-(\ell+j)}U) \cap X_{n_j, 0} \subseteq T^{-(\ell+j)}(U \cap T^{\ell+j}X_{n_j, 0}) = T^{-(\ell+j)}U_{n_j, i_{n_j}} = \emptyset.$$

Thus, for all $1 \leq j \leq h$, $(T^{-(\ell+j)}U) \cap X_{n_j, 0} = \emptyset$.

On the other hand, since d_{n_1}, \dots, d_{n_h} are pairwise coprime, again by the Chinese Remainder Theorem,

$$\bigcup_{n=1}^{\infty} T^{-n} \bigcap_{j=1}^h X_{n_j, 0} = \bigcup_{n=1}^{\infty} \bigcap_{j=1}^h X_{n_j, -n} = X,$$

which implies that $\bigcap_{j=1}^h X_{n_j, 0} \neq \emptyset$. Since $\bigcup_{j=1}^h T^{-j}U = X$, $\bigcup_{j=1}^h T^{-(\ell+j)}U = X$. Putting these facts together, we see

$$\bigcap_{j=1}^h X_{n_j, 0} = \left(\bigcup_{j=1}^h T^{-(\ell+j)}U \right) \cap \left(\bigcap_{j=1}^h X_{n_j, 0} \right) \subseteq \bigcup_{j=1}^h \left((T^{-(\ell+j)}U) \cap X_{n_j, 0} \right),$$

a contradiction since the leftmost set was shown to be non-empty while the rightmost set was shown to be empty. \square

Proposition 3.2. *Let (X, T) be a minimal system. For all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $(n, N) = 1$ and all $x \in X$, the T^n -orbit of x is ε -dense in X .*

Proof. Let $\varepsilon > 0$. Since X is compact, there exist $x_1, \dots, x_h \in X$ for which $X = \bigcup_{i=1}^h B(x_i, \varepsilon/2)$. For each $1 \leq i \leq h$, Lemma 3.1 gives the existence of $N_i \in \mathbb{N}$ such that for all $n \in \mathbb{N}$ with $(n, N_i) = 1$ and all $x \in X$, $\bar{o}_{T^n}(x) \cap B(x_i, \varepsilon/2) \neq \emptyset$. We claim that $N := \prod_{i=1}^h N_i$ has the required property.

Let $n \in \mathbb{N}$ with $(n, N) = 1$, $x \in X$, and $1 \leq i \leq h$. Since $(n, N_i) = 1$, Lemma 3.1 gives that $\bar{o}_{T^n}(x) \cap B(x_i, \varepsilon/2) \neq \emptyset$. Since $1 \leq i \leq h$ was arbitrary and $X = \bigcup_{i=1}^h B(x_i, \varepsilon/2)$, the orbit $\bar{o}_{T^n}(x)$ is ε -dense. \square

The family $\{\mathbb{Z}/d_n\mathbb{Z} \mid n \in \mathbb{N}\}$, directed via the maps $\mathbb{Z}/d_{nm}\mathbb{Z} \rightarrow \mathbb{Z}/d_n\mathbb{Z}$, gives rise to the *rational topological Kronecker factor* of (X, T) : the system

$$\mathbf{K}_{\text{rat}}(X, T) := (Z := \varprojlim_{n \in \mathbb{N}} \mathbb{Z}/d_n\mathbb{Z}, T),$$

where $T : (a_n)_{n \in \mathbb{N}} \mapsto (a_n + 1)_{n \in \mathbb{N}}$ is a minimal rotation of the compact abelian group Z . Defining $Z_{n,i}$ just as it was defined for X at the beginning of this section, we see that $d_n(Z, T) = d_n(X, T)$ and, by the topology on Z , that the factor map $\pi : (X, T) \rightarrow (Z, T)$ is defined uniquely by the property $\pi(X_{n,i}) = Z_{n,i}$. Also, note that for any non-empty, open set $V \subseteq Z$, there exists $n, i \in \mathbb{N}$ such that $Z_{n,i} \subseteq V$.

The goal for the remainder of this section is to prove Lemma 3.5, a result related to Lemma 3.1 on the measure of the sets $U \cap X_{N,j}$ in distal systems. This will be accomplished with the help of the topological Kronecker factor of (X, T) .

Definition 3.3. Let (X, T) be a system, $U \subseteq X$ be open, non-empty, and μ be a T -invariant probability measure on X . The set U is *totally visible by μ* if

$$\inf_{n, i \in \mathbb{N}} \mu_{n,i}(U) > 0.$$

The set U is *totally visible* if it is totally visible by some T -invariant probability measure μ on X .

Remark 3.4. If (X, T) is totally minimal, then for all $n, i \in \mathbb{N}$, $X_{n,i} = X$ and $\mathbf{K}_{\text{rat}}(X, T)$ is trivial. For any T -invariant measure μ and all $n, i \in \mathbb{N}$, $\mu_{n,i} = \mu$. It follows that in totally minimal systems, all open sets are totally visible by any invariant probability measure.

Lemma 3.5. *Let (X, T) be a minimal, distal system, and let $U \subseteq X$ be a non-empty, open set. There exists $N, j \in \mathbb{N}$ so that $U_{N,j}$ is totally visible in the system $(X_{N,j}, T^N)$.*

Proof. Let $W \subseteq V \subseteq U$ and $\varepsilon > 0$ be such that W and V are non-empty and open, $\bar{V} \subseteq U$, and for all $x \in W$, $B(x, 2\varepsilon) \subseteq \bar{V}$. Put $(Z, T) = \mathbf{K}_{\text{rat}}(X, T)$. By [Fur1, Theorem 8.1], the factor map $\pi : X \rightarrow Z$ is open, so there exists $N, j \in \mathbb{N}$ such that $Z_{N,j} \subseteq \pi W$.

Let $z_0 \in Z$. By [Fur1, Lemma 8.1], there exists a finite set $F \subseteq \pi^{-1}(\{z_0\})$ such that for all $n \in \mathbb{Z}$, the set $T^n F$ is ε -dense in the fiber $\pi^{-1}(\{T^n z_0\})$.

Put $\eta = (2|F|)^{-1}$, and let

$$\nu = \frac{1}{|F|} \sum_{f \in F} \delta_f.$$

We claim that for all $n, i \in \mathbb{N}$ and all $k \in \mathbb{Z}$,

$$(8) \quad T^k \nu(\bar{V} \cap X_{nN, Ni+j}) \geq \eta T^k \delta_{z_0}(Z_{nN, Ni+j}).$$

To see why, note that the right hand side is zero unless $T^k z_0 \in Z_{nN, Ni+j}$. Suppose that $T^k z_0 \in Z_{nN, Ni+j}$. Because $Z_{nN, Ni+j} \subseteq Z_{N, j} \subseteq \pi W$, there exists a point $x \in \pi^{-1}(\{T^k z_0\}) \cap W$. Since $x \in W$, $B(x, 2\varepsilon) \subseteq \bar{V}$. Because $T^k F$ is ε -dense in $\pi^{-1}(\{T^k z_0\})$, at least one point of $T^k F$ is in \bar{V} . This combined with the fact that $T^k \nu$ is supported on $\pi^{-1}(\{T^k z_0\}) \subseteq X_{nN, Ni+j}$ implies that $T^k \nu(\bar{V} \cap X_{nN, Ni+j}) \geq \eta$, showing (8).

Let μ be a weak- $*$ limit point of the set $\{N^{-1} \sum_{k=0}^{N-1} T^k \nu \mid N \in \mathbb{N}\}$. We claim that the set $U_{N, j}$ is visible by the measure $\mu_{N, j}$ in the system $(X_{N, j}, T^N)$. We must show that for all $n, i \in \mathbb{N}$,

$$(9) \quad (\mu_{N, j})_{n, i}(U_{N, j}) \geq \eta.$$

Let $n, i \in \mathbb{N}$, and recall that $d_n = d_n(X, T) = d_n(Z, T)$. Using (8), we see

$$\begin{aligned} (\mu_{N, j})_{n, i}(U_{N, j}) &= \mu_{nN, Ni+j}(U) \\ &= d_{nN} \mu(U \cap X_{nN, Ni+j}) \\ &\geq d_{nN} \mu(\bar{V} \cap X_{nN, Ni+j}) \\ &\geq d_{nN} \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} T^k \nu(\bar{V} \cap X_{nN, Ni+j}) \\ &\geq d_{nN} \eta \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} T^k \delta_{z_0}(Z_{nN, Ni+j}) = \eta, \end{aligned}$$

where the last equality follows from the ergodic theorem because z_0 is generic for the Haar measure on Z and the Haar measure of $Z_{nN, Ni+j}$ is $1/d_{nN}$. This establishes (9), concluding the proof. \square

4. DYNAMICS ON THE ORBIT CLOSURE OF THE DIAGONAL

Suppose (X, T) is minimal and invertible, and fix $\ell \in \mathbb{N}$ and $\vec{m} \in \mathbb{N}^\ell$. Put $M = \text{LCM}(\vec{m})$, and let $d_M = d_M(X, T)$ be as described in Section 3. We will now prove some preliminary results concerning dynamics of points along the diagonal of X^ℓ and points of continuity of the orbit closure map $\bar{o}_{T^{m_1} \times \dots \times T^{m_\ell}}$.

Let $\Delta : X \rightarrow X^\ell$, $x \mapsto (x, \dots, x)$, be the diagonal injection. Let

$$\begin{aligned} \Delta(T) &:= T \times \dots \times T, \\ T^{\vec{m}} &:= T^{m_1} \times \dots \times T^{m_\ell}, \\ X^\Delta &:= \bar{o}_{T^{\vec{m}}}(\Delta(X)) := \overline{\bigcup_{n \in \mathbb{N}_0} (T^{\vec{m}})^n \Delta(X)} \subseteq X^\ell, \\ X_{M, j}^\Delta &:= \bar{o}_{T^{\vec{m}}}(\Delta(X_{M, j})) \subseteq X^\Delta, \quad j \in \{0, \dots, d_M - 1\}. \end{aligned}$$

Note that because T is a homeomorphism, $\Delta(T)$ and $T^{\vec{m}}$ are commuting homeomorphisms of X^ℓ .

Theorem 4.1. *The maps $\Delta(T)$ and $T^{\vec{m}}$ are homeomorphisms of X^Δ , and the system $(X^\Delta, \Delta(T), T^{\vec{m}})$ is minimal.*

Proof. Let

$$(10) \quad \mathbb{X}^\Delta := \overline{\bigcup_{n \in \mathbb{Z}} (T^{\vec{m}})^n \Delta(X)} \subseteq X^\ell.$$

It is immediate that $\Delta(T)$ and $T^{\vec{m}}$ are homeomorphisms of \mathbb{X}^Δ . It is proved in [Gla1, Theorem 5.1] that the system $(\mathbb{X}^\Delta, \Delta(T), T^{\vec{m}})$ is minimal in the case that $\vec{m} = (1, 2, \dots, \ell)$. Since factors of minimal systems are minimal, and since $(\mathbb{X}^\Delta, \Delta(T), T^{\vec{m}})$ is a factor of a system to which Glasner's theorem applies (for example, the one corresponding to the vector $(1, 2, \dots, \max_i \vec{m}_i)$), it is minimal.

Let $\vec{x} \in X^\Delta \subseteq \mathbb{X}^\Delta$. Since \mathbb{X}^Δ is minimal, by Lemma 2.7,

$$\mathbb{X}^\Delta = \overline{\{\Delta(T)^n (T^{\vec{m}})^k \vec{x} \mid n, k \in \mathbb{N}\}} \subseteq X^\Delta.$$

This shows that $X^\Delta = \mathbb{X}^\Delta$. Therefore, $\Delta(T)$ and $T^{\vec{m}}$ are homeomorphisms of X^Δ and $(X^\Delta, \Delta(T), T^{\vec{m}})$ is minimal. \square

Theorem 4.2. *The $X_{M,j}^\Delta$'s are mutually disjoint, clopen, and*

$$X^\Delta = X_{M,0}^\Delta \cup \dots \cup X_{M,d_M-1}^\Delta.$$

The maps $\Delta(T)^M$ and $T^{\vec{m}}$ are homeomorphisms of $X_{M,j}^\Delta$, and the system $(X_{M,j}^\Delta, \Delta(T)^M, T^{\vec{m}})$ is minimal.

Proof. Since $X = \bigcup_{j=0}^{d_M-1} X_{M,j}$, it follows immediately from the definition of X^Δ that

$$X^\Delta = X_{M,0}^\Delta \cup \dots \cup X_{M,d_M-1}^\Delta.$$

We will show next that the $X_{M,j}^\Delta$'s are mutually disjoint. Since they are closed, disjointness will imply that the $X_{M,j}^\Delta$'s are open, hence clopen.

Suppose $j, j' \in \{0, \dots, d_M - 1\}$ are such that $X_{M,j}^\Delta \cap X_{M,j'}^\Delta \neq \emptyset$; we will show that $j = j'$. Let $\vec{x} = (x_1, \dots, x_\ell) \in X_{M,j}^\Delta \cap X_{M,j'}^\Delta$. By the definition of $X_{M,j}^\Delta$, there exist sequences $(n_k)_{k \in \mathbb{N}}, (n'_k)_{k \in \mathbb{N}} \subseteq \mathbb{N}$, $(y_k)_{k \in \mathbb{N}} \subseteq X_{M,j}$, and $(y'_k)_{k \in \mathbb{N}} \subseteq X_{M,j'}$ so that

$$\lim_{k \rightarrow \infty} (T^{\vec{m}})^{n_k} \Delta(y_k) = \lim_{k \rightarrow \infty} (T^{\vec{m}})^{n'_k} \Delta(y'_k) = \vec{x}.$$

It follows that for each $i \in \{1, \dots, \ell\}$,

$$\lim_{k \rightarrow \infty} T^{m_i n_k} y_k = \lim_{k \rightarrow \infty} T^{m_i n'_k} y'_k = x_i.$$

For each $i \in \{1, \dots, \ell\}$, let $j_i \in \{0, \dots, d_M - 1\}$ be such that $x_i \in X_{M,j_i}$. Since $T^{m_i n_k} y_k \in X_{M,j+m_i n_k}$ and $x_i \in X_{M,j_i}$, it must be that $j + m_i n_k \equiv j_i \pmod{d_M}$ for all sufficiently large k . Similarly, we can conclude that $j' + m_i n'_k \equiv j_i \pmod{d_M}$, meaning $m_i(n'_k - n_k) \equiv j - j' \pmod{d_M}$ for all sufficiently large k . This implies that $j - j'$ is a multiple of (m_i, d_M) for all $i \in \{1, \dots, \ell\}$, whereby (M, d_M) divides $j - j'$. Since $(M, d_M) = d_M$, it follows that $j \equiv j' \pmod{d_M}$, implying $j = j'$.

For $j \in \{0, \dots, d_M - 1\}$, let

$$\mathbb{X}_{M,j}^\Delta := \overline{\bigcup_{n \in \mathbb{Z}} (T^{\vec{m}})^n \Delta(X_{M,j})} \subseteq X^\Delta.$$

Since $TX_{M,j} = X_{M,j+1}$, we have that $\Delta(T)^M$ and $T^{\vec{m}}$ are homeomorphisms of $\mathbb{X}_{M,j}^\Delta$. It also follows that $\Delta(T)\mathbb{X}_{M,j}^\Delta = \mathbb{X}_{M,j+1}^\Delta$ and that

$$\mathbb{X}^\Delta = \mathbb{X}_{M,0}^\Delta \cup \dots \cup \mathbb{X}_{M,d_M-1}^\Delta,$$

where \mathbb{X}^Δ is as defined in (10). It was shown in that proof that $X^\Delta = \mathbb{X}^\Delta$; combining this with the facts that $X_{M,j}^\Delta \subseteq \mathbb{X}_{M,j}^\Delta$ and that the $X_{M,j}^\Delta$'s are mutually

disjoint, we see $\mathbb{X}_{M,j}^\Delta = X_{M,j}^\Delta$. This shows that $\Delta(T)^M$ and $T^{\bar{m}}$ are homeomorphisms of $X_{M,j}^\Delta$.

To show that $(X_{M,j}^\Delta, \Delta(T)^M, T^{\bar{m}})$ is minimal, we will show that every point has a dense orbit, starting with points on the diagonal. Let $x \in X_{M,j}$ and consider

$$Y := \overline{\{\Delta(T)^{Mn}(T^{\bar{m}})^k \Delta(x) \mid n, k \in \mathbb{N}_0\}} \subseteq X_{M,j}^\Delta.$$

Since $(X_{M,j}, T^M)$ is minimal, $\Delta(X_{M,j}) \subseteq Y$, and, moreover, for all $k \in \mathbb{N}$, $(T^{\bar{m}})^k \Delta(X_{M,j}) \subseteq Y$. Since Y is closed, it follows that $X_{M,j}^\Delta \subseteq Y$, which implies that $Y = X_{M,j}^\Delta$. Thus, points on the diagonal have a dense orbit.

Let $\vec{x} \in X_{M,j}^\Delta$, and let Y be the $(\Delta(T)^M, T^{\bar{m}})$ -orbit closure of \vec{x} . Let $w \in X_{M,j}$, and note that $\Delta(w) \in X_{M,j}^\Delta$. By Theorem 4.1, the system $(X^\Delta, \Delta(T), T^{\bar{m}})$ is minimal, so there exists a sequence $((a_n, b_n))_{n \in \mathbb{N}} \subseteq \mathbb{N}^2$ for which $(T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \vec{x} \rightarrow \Delta(w)$ as $n \rightarrow \infty$. By passing to a subsequence, we may assume that there exists $0 \leq b \leq M-1$ such that for all $n \in \mathbb{N}$, $b_n \equiv b \pmod{M}$. Since $((T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \vec{x})_{n \in \mathbb{N}} \subseteq X_{M,j+b}^\Delta$, $\Delta(w) \in X_{M,j+b}^\Delta$. Since the $X_{M,j}^\Delta$'s are disjoint and $\Delta(w) \in X_{M,j}^\Delta \cap X_{M,j+b}^\Delta$, $b = 0$, whereby $\Delta(w) \in Y$. It follows now from the previous paragraph that $Y = X_{M,j}^\Delta$. \square

Lemma 4.3. *For all open, non-empty $U \subseteq X_{M,j}$, the set*

$$X_U^\Delta := \bar{o}_{T^{\bar{m}}}(\Delta(U))$$

has non-empty interior in $X_{M,j}^\Delta$.

Proof. Let $U \subseteq X_{M,j}$ be open, non-empty. Since $(X_{M,j}, T^M)$ is minimal, there exists $h \in \mathbb{N}$ such that

$$X_{M,j}^\Delta = \bigcup_{i=1}^h X_{T^{-Mi}U}^\Delta = \bigcup_{i=1}^h \Delta(T)^{-Mi} X_U^\Delta.$$

Since $X_{M,j}^\Delta$ is a Baire space (it is a compact metric space), some $\Delta(T)^{-Mi} X_U^\Delta$ has non-empty interior. Since $\Delta(T)^M$ is a homeomorphism, it is open, implying that X_U^Δ has non-empty interior. \square

Proposition 4.4. *Let Ω be the set of points of continuity of the map $\bar{o}_{T^{\bar{m}}} : X^\Delta \rightarrow \mathcal{F}(X^\Delta)$. The set $\Omega \cap \Delta(X)$ is a residual subset of $\Delta(X)$. For $j \in \{0, \dots, d_M - 1\}$, the set $\Omega_{M,j} := \Omega \cap X_{M,j}^\Delta$ is the set of points of continuity of the map $\bar{o}_{T^{\bar{m}}} : X_{M,j}^\Delta \rightarrow \mathcal{F}(X_{M,j}^\Delta)$ and $\Omega_{M,j} \cap \Delta(X_{M,j})$ is a residual subset of $\Delta(X_{M,j})$.*

Proof. Let $\Xi = X^\Delta \setminus \Omega$ be the set of points of discontinuity of the map $\bar{o}_{T^{\bar{m}}} : X^\Delta \rightarrow \mathcal{F}(X^\Delta)$. By Lemma 2.14, there exists a countable family $\{B_i\}_{i \in \mathbb{N}}$ of closed, $T^{\bar{m}}$ -invariant, empty-interior subsets of X^Δ for which $\Xi \subseteq \bigcup_i B_i$. We claim that each $B_i \cap \Delta(X)$ is a closed set with empty interior in $\Delta(X)$. It is closed because $\Delta(X)$ is closed. Suppose for a contradiction that $U \subseteq X$ is open and is such that $\Delta(U) \subseteq B_i$. Since B_i is $T^{\bar{m}}$ -invariant and closed, $X_U^\Delta \subseteq B_i$. It follows by Lemma 4.3 that B_i has non-empty interior, a contradiction.

Note that $\Delta(X) \cap \Xi \subseteq \bigcup_i (B_i \cap \Delta(X))$ is a cover of $\Delta(X) \cap \Xi$ with closed sets with empty interior, meaning $\Delta(X) \cap \Xi$ is a meager subset of $\Delta(X)$. Since $\Delta(X) = (\Delta(X) \cap \Omega) \cup (\Delta(X) \cap \Xi)$, the set $\Delta(X) \cap \Omega$ is a residual subset of $\Delta(X)$.

Let $j \in \{0, \dots, d_M - 1\}$. Since $\bar{o}_{T^{\bar{m}}} : X_{M,j}^\Delta \rightarrow \mathcal{F}(X_{M,j}^\Delta)$ is the restriction of the map $\bar{o}_{T^{\bar{m}}} : X^\Delta \rightarrow \mathcal{F}(X^\Delta)$ to $X_{M,j}^\Delta$, the set of its points of continuity is $\Omega_{M,j} =$

$\Omega \cap X_{M,j}^\Delta$. Since $\Delta(X_{M,j})$ is an open subset of $\Delta(X)$ and Ω is residual in $\Delta(X)$, the set $\Omega \cap \Delta(X_{M,j}) = \Omega_{M,j} \cap \Delta(X_{M,j})$ is a residual subset of $\Delta(X_{M,j})$. \square

5. RESULTS ON MINIMAL SYSTEMS

We begin this section by demonstrating the equivalence between Theorems 1.7 and 1.8, dynamical formulations of van der Waerden's theorem from the Introduction.

Proof of equivalence between Theorems 1.7 and 1.8. A point $x \in X$ is called *multiply recurrent* (cf. [Fur3, page 9]) if for all $k \in \mathbb{N}$ and all $\varepsilon > 0$ there exists $n \in \mathbb{N}$ such that $\max_{i=1,\dots,k} d(x, T^{in}x) < \varepsilon$. Let X' be the set of multiply recurrent points in X ; by [Fur3, Theorem MBR, page 9], the set X' is non-empty. We claim that X' is a dense G_δ subset of X and that for any point $x \in X'$ and for any non-empty, open $U \subseteq X$ containing x , the set $R(x, U)$ satisfies $d_\times^*(R(x, U)) = 1$.

First we will show that X' is dense G_δ subset of X . Since (X, T) is minimal, any non-empty, T -invariant subset, such as X' , is dense. To show that X' is a G_δ subset of X , let $(x_j)_{j \in \mathbb{N}} \subseteq X$ be dense in X . We claim that

$$(11) \quad X' = \bigcap_{(m,k) \in \mathbb{N}^2} \bigcup_{(n,j) \in \mathbb{N}^2} \bigcap_{i=0}^k T^{-in} B\left(x_j, \frac{1}{m}\right),$$

which will show that X' is a G_δ set. To see the equality, note that if $x \in X'$, then for any $m, k \in \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $\max_{i=1,\dots,k} d(x, T^{in}x) < 1/2m$. If $j \in \mathbb{N}$ is such that $d(x, x_j) < 1/2m$, then $\max_{i=0,\dots,k} d(T^{in}x, x_j) < 1/m$. On the other hand, if x is contained in the right-hand side of (11), $k \in \mathbb{N}$, and $\varepsilon > 0$, then let $m \in \mathbb{N}$ be such that $2/m < \varepsilon$. There exists $n, j \in \mathbb{N}$ such that $\max_{i=0,\dots,k} d(T^{in}x, x_j) < 1/m$, which implies that $\max_{i=1,\dots,k} d(x, T^{in}x) < 2/m < \varepsilon$. It follows that $x \in X'$.

Finally, we claim that if x is a multiply recurrent point and U is a non-empty, open neighborhood of x , then $d_\times^*(R(x, U)) = 1$. Equivalently, we must show that for all finite $F \subseteq \mathbb{N}$, there exists $n \in \mathbb{N}$ such that $nF \subseteq R(x, U)$. Let $k \in \mathbb{N}$ be such that $F \subseteq \{1, \dots, k\}$, and let $\varepsilon > 0$ be such that $B(x, \varepsilon) \subseteq U$. Since x is multiply recurrent, there exists $n \in \mathbb{N}$ such that for all $i \in \{1, \dots, k\}$, $d(x, T^{in}x) < \varepsilon$. This means that $\{n, 2n, \dots, kn\} \subseteq R(x, B(x, \varepsilon)) \subseteq R(x, U)$, whereby $nF \subseteq R(x, U)$, as desired. \square

Let (X, T) be a topological dynamical system, $x \in X$, and $U \subseteq X$ be non-empty, open. Finding a configuration of the form $\{n, nm, nm^2\}$ in $R(x, U)$ is equivalent to showing that the $T \times T^m \times T^{m^2}$ -orbit closure of (x, x, x) in X^3 has non-empty intersection with $U \times U \times U$. This observation motivates the approach we use in the proof of Theorem 1.1.

The first step in the proof is to show that for any $\ell \in \mathbb{N}$ and $\vec{m} \in \mathbb{N}^\ell$, the $T^{\vec{m}}$ -orbit closure of many points $\Delta(x) = (x, \dots, x)$ along the diagonal in X^Δ supports a measure ν_x^Δ whose marginals on X give mass $\eta > 0$ to U . The second step is to use Lemma 2.11 to find a mean with respect to which the point $\Delta(x)$ is ν_x^Δ -generic. Because each coordinate of $\Delta(x)$ spends an η -proportion of time in U under $T^{\vec{m}}$, there must be many times for which an η -proportion of the coordinates are simultaneously in U . Szemerédi's theorem then allows us to finish the argument by taking \vec{m} to be a sufficiently long geometric progression.

Proposition 5.1. *Let (X, T) be an invertible, minimal dynamical system, and let $U \subseteq X$ be open, non-empty. There exists $\eta > 0$ such that for all $\varepsilon > 0$, there exists $N \in \mathbb{N}$ such that for all finite sets $F \subseteq S_N$, there exists an ε -dense subset $X_\varepsilon \subseteq X$ such that for all $x \in X_\varepsilon$, there exists $F' \subseteq F$ with $|F'| > \eta|F|$ and $n \in \mathbb{N}$ such that $nF' \subseteq R(x, U)$.*

Proof. Let $V \subseteq X$ open, non-empty such that $\bar{V} \subseteq U$, and let μ be any T -invariant probability measure on X . Since T is minimal, $\eta := (\mu(V)/2)^2 > 0$. Let $\varepsilon > 0$, and let $N \in \mathbb{N}$ be as given in Proposition 3.2.

Let $F = \{m_1, \dots, m_\ell\} \subseteq S_N$, and put $\bar{m} = (m_1, \dots, m_\ell) \in S_N^\ell$. Let

$$X^\Delta := \bar{o}_{T^{\bar{m}}}(\Delta(X)).$$

Let μ^Δ be any weak-* limit point of the set $\{N^{-1} \sum_{n=0}^{N-1} (T^{\bar{m}})^n \Delta(\mu) \mid N \in \mathbb{N}\}$, where $\Delta(\mu)$ denotes the push-forward of μ under the map Δ . It follows that μ^Δ is a $T^{\bar{m}}$ -invariant probability measure on X^Δ with marginals $\pi_i \mu^\Delta = \mu$, where $\pi_i : X^\Delta \rightarrow X$ is the projection onto the i^{th} coordinate.

By Lemma 2.13, the map $\bar{o}_{T^{\bar{m}}} : X^\Delta \rightarrow \mathcal{F}(X^\Delta)$ is lower semicontinuous, hence Borel. Denote by \mathcal{B}_{X^Δ} and $\mathcal{B}_{\mathcal{F}(X^\Delta)}$ the Borel σ -algebras of X^Δ and $\mathcal{F}(X^\Delta)$, respectively. Let \mathcal{A} be the pull-back of $\mathcal{B}_{\mathcal{F}(X^\Delta)}$ through $\bar{o}_{T^{\bar{m}}}$. Because $\bar{o}_{T^{\bar{m}}}$ is Borel, \mathcal{A} is a sub- σ -algebra of \mathcal{B}_{X^Δ} , and because $\mathcal{B}_{\mathcal{F}(X^\Delta)}$ is countably generated, so too is \mathcal{A} . Note that $\mathfrak{a}(\vec{x}) := \bar{o}_{T^{\bar{m}}}^{-1}(\{\bar{o}_{T^{\bar{m}}}(\vec{x})\}) \in \mathcal{A}$ is the atom of \mathcal{A} containing $\vec{x} \in X^\Delta$.

Disintegrating μ^Δ with respect to \mathcal{A} (see, e.g., [EW, Theorem 5.14]), there exists a μ^Δ -co-null set $X_0^\Delta \subseteq X^\Delta$ and, for each $\vec{x} \in X_0^\Delta$, a Borel probability measure $\mu_{\vec{x}}^\Delta$ supported on $\mathfrak{a}(\vec{x}) \subseteq \bar{o}_{T^{\bar{m}}}(\vec{x})$ such that $\mu_{\vec{x}}^\Delta = \mu_{\vec{y}}^\Delta$ whenever $\mathfrak{a}(\vec{x}) = \mathfrak{a}(\vec{y})$ and such that

$$\mu^\Delta = \int_{X_0^\Delta} \mu_{\vec{x}}^\Delta d\mu^\Delta(\vec{x}).$$

By the essential uniqueness of this disintegration, the $T^{\bar{m}}$ -invariance of μ^Δ , and the fact that $T^{\bar{m}}\mathfrak{a}(\vec{x}) = \mathfrak{a}(T^{\bar{m}}\vec{x})$, it follows that $T^{\bar{m}}\mu_{\vec{x}}^\Delta = \mu_{T^{\bar{m}}\vec{x}}^\Delta$. Note that if \vec{x} is a $T^{\bar{m}}$ -recurrent point, then $\bar{o}_{T^{\bar{m}}}(\vec{x}) = \bar{o}_{T^{\bar{m}}}(T^{\bar{m}}\vec{x})$. For such points, $\mathfrak{a}(\vec{x}) = \mathfrak{a}(T^{\bar{m}}\vec{x})$, whereby $T^{\bar{m}}\mu_{\vec{x}}^\Delta = \mu_{\vec{x}}^\Delta$, meaning $\mu_{\vec{x}}^\Delta$ is $T^{\bar{m}}$ -invariant. By [BS, Proposition 4.2.2], μ^Δ -almost every point $\vec{x} \in X^\Delta$ is $T^{\bar{m}}$ -recurrent, so by passing to a μ^Δ -co-null subset of X_0^Δ , we may assume that for all $\vec{x} \in X_0^\Delta$, the measure $\mu_{\vec{x}}^\Delta$ is $T^{\bar{m}}$ -invariant.

For $i \in \{1, \dots, \ell\}$, let

$$X_i^\Delta = \{\vec{x} \in X_0^\Delta \mid \pi_i \mu_{\vec{x}}^\Delta(\bar{V}) > \sqrt{\eta}\}.$$

Since

$$2\sqrt{\eta} < \mu(\bar{V}) = \pi_i \mu^\Delta(\bar{V}) = \int_{X_0^\Delta} \pi_i \mu_{\vec{x}}^\Delta(\bar{V}) d\mu^\Delta(\vec{x}),$$

we have by Chebyshev's inequality that $\mu^\Delta(X_i^\Delta) > \sqrt{\eta}$. By the pigeonhole principle (with reasoning similar to that in the proof of Lemma 2.1), there exists $I \subseteq \{1, \dots, \ell\}$ with $|I| > \sqrt{\eta}\ell$ for which $X_0^\Delta \cap \bigcap_{i \in I} X_i^\Delta \neq \emptyset$. Let \vec{w} be an element of this set.

Put $M = \text{LCM}(\bar{m})$. By Proposition 4.4, there exists a point $x \in X$ for which $\vec{x} := \Delta(x)$ is a point of continuity of the map $\bar{o}_{T^{\bar{m}}}$. By Theorem 4.1, there exists a sequence $((a_n, b_n))_{n \in \mathbb{N}} \subseteq \mathbb{N}^2$ for which $(T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \vec{w} \rightarrow \vec{x}$ as $n \rightarrow \infty$. By passing to a subsequence if necessary, there exists $0 \leq b \leq M - 1$ such that for all $n \in \mathbb{N}$, $b_n \equiv b \pmod{M}$. By Proposition 4.4, the points of continuity of the

map $\bar{o}_{T^{\bar{m}}}$ are $\Delta(T)$ -invariant. Therefore, by replacing x with $T^{M-b}x$ and b_n with $b_n + M - b$, we may assume that $b = 0$, that is, that $(b_n)_{n \in \mathbb{N}} \subseteq M\mathbb{N}$.

Let ν_x^Δ be a weak-* limit point of the set $\{\Delta(T)^{b_n} \mu_{\bar{w}}^\Delta \mid n \in \mathbb{N}\}$; by passing to a subsequence, we may assume without loss of generality that $\Delta(T)^{b_n} \mu_{\bar{w}}^\Delta \rightarrow \nu_x^\Delta$ as $n \rightarrow \infty$. We will show now that ν_x^Δ is a $T^{\bar{m}}$ -invariant probability measure supported on $x^\Delta := \bar{o}_{T^{\bar{m}}}(\Delta(x))$ such that for all $i \in I$, $\pi_i \nu_x^\Delta(\bar{V}) > \sqrt{\eta}$.

That ν_x^Δ is a probability measure is immediate from its definition. The measure ν_x^Δ is $T^{\bar{m}}$ -invariant because $\Delta(T)$ and $T^{\bar{m}}$ commute, are continuous, and $\mu_{\bar{w}}^\Delta$ is $T^{\bar{m}}$ -invariant. Since $(T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \bar{w} \rightarrow \bar{x}$ and \bar{x} is a point of continuity of $\bar{o}_{T^{\bar{m}}}$, $\bar{o}_{T^{\bar{m}}}((T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \bar{w}) \rightarrow x^\Delta$ as $n \rightarrow \infty$. Combined with the fact that $\text{supp}((T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \mu_{\bar{w}}^\Delta) \subseteq \bar{o}_{T^{\bar{m}}}((T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \bar{w})$, Lemma 2.15 gives that the measure ν_x^Δ is supported on x^Δ .

Let $i \in I$. Because $\mu_{\bar{w}}^\Delta$ is $T^{\bar{m}}$ -invariant, the measure $\pi_i \mu_{\bar{w}}^\Delta$ is T^{m_i} -invariant, and hence, for all $n \in \mathbb{N}$, T^{b_n} -invariant. By properties of weak-* convergence of measures and this invariance,

$$\begin{aligned} \pi_i \nu_x^\Delta(\bar{V}) &\geq \pi_i \limsup_{n \rightarrow \infty} (T^{\bar{m}})^{a_n} \Delta(T)^{b_n} \mu_{\bar{w}}^\Delta(\bar{V}) \\ (12) \quad &= \limsup_{n \rightarrow \infty} T^{b_n} \pi_i \mu_{\bar{w}}^\Delta(\bar{V}) \\ &= \pi_i \mu_{\bar{w}}^\Delta(\bar{V}) > \sqrt{\eta}. \end{aligned}$$

Applied to the system $(x^\Delta, T^{\bar{m}}, \nu_x^\Delta)$, Lemma 2.11 gives the existence of an additively invariant mean λ on \mathbb{N} such that for all $g \in C(x^\Delta)$,

$$\lambda(n \mapsto g(T^{m_1 n} x, T^{m_2 n} x, \dots, T^{m_{\ell} n} x)) = \int_{x^\Delta} g d\nu_x^\Delta.$$

By Urysohn's lemma, there exists a continuous function $f : X \rightarrow [0, 1]$ with $f(y) = 1$ for all $y \in \bar{V}$ and $f(y) = 0$ for all $y \notin U$. By (12), for all $i \in I$,

$$\begin{aligned} \lambda(R(x, U)/m_i) &\geq \lambda(n \mapsto f(T^{m_i n} x)) \\ &= \lambda(n \mapsto (f \circ \pi_i)(T^{m_1 n} x, T^{m_2 n} x, \dots, T^{m_{\ell} n} x)) \\ &= \int_{x^\Delta} f \circ \pi_i d\nu_x^\Delta \\ &\geq \pi_i \nu_x^\Delta(\bar{V}) > \sqrt{\eta}. \end{aligned}$$

Put $X_\varepsilon = \{T^{Mk} x \mid k \in \mathbb{N}\}$. Since $M \in S_N$, by Proposition 3.2, the set X_ε is ε -dense in X . Therefore, to conclude the proof, it suffices to show that for every $k \in \mathbb{N}$, there exists $F' \subseteq F$ with $|F'| > \eta|F|$ and $n \in \mathbb{N}$ such that $nF' \subseteq R(T^{Mk} x, U)$.

Let $k \in \mathbb{N}$. By the translation invariance of λ , for all $i \in I$,

$$\lambda\left(\frac{R(T^{Mk} x, U)}{m_i}\right) = \lambda\left(\frac{R(x, U) - Mk}{m_i}\right) = \lambda\left(\frac{R(x, U)}{m_i} - \frac{M}{m_i} k\right) > \sqrt{\eta}.$$

It follows by Lemma 2.1 that there exists $I' \subseteq I$ with $|I'| > \sqrt{\eta}|I|$ and $n \in \mathbb{N}$ such that for all $i \in I'$, $m_i n \in R(T^{Mk} x, U)$. Setting $F' = \{m_i \mid i \in I'\}$, we see $|F'| > \eta|F|$ and $nF' \subseteq R(T^{Mk} x, U)$, as was to be shown. \square

Now we can prove Theorem 1.1, restated here.

Theorem 1.1. *Let (X, T) be a minimal dynamical system. There exists a residual set $X' \subseteq X$ such that for all $x \in X'$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ contains arbitrarily long geometric progressions.*

Proof. It suffices to prove the statement for (X, T) invertible. To see why, let $\pi : (W, T) \rightarrow (X, T)$ be the natural extension (Definition 2.8). Suppose that Theorem 1.1 holds for (W, T) : there exists a residual set $W' \subseteq W$ such that for all $w \in W'$ and all non-empty, open $V \subseteq X$, the set $R(w, V)$ contains arbitrarily long geometric progressions. By Lemmas 2.6 and 2.9, the set $X' := \pi W'$ is residual. Let $x \in X'$ and $U \subseteq X$ open, non-empty. Choose $w \in \pi^{-1}(\{x\}) \cap W'$, and note that $R(w, \pi^{-1}U) = R(x, U)$. It follows that $R(x, U)$ contains arbitrarily long geometric progressions.

By taking a countable basis of open sets and a countable intersection of residual sets, the residual set X' is allowed to depend on the set U . Let $U \subseteq X$ be open, non-empty. Let $\eta > 0$ be as guaranteed by Proposition 5.1 for the set U .

For $\ell \in \mathbb{N}$, put

$$G_\ell := \bigcup_{\substack{m, n \in \mathbb{N} \\ m \geq 2}} (T^{-nm}U \cap \dots \cap T^{-nm^\ell}U).$$

This is precisely the set of points $x \in X$ for which $R(x, U)$ contains a geometric progression of length ℓ . The set of those points $x \in X$ for which $R(x, U)$ contains arbitrarily long geometric progressions is thus $X' := \bigcap_{\ell \in \mathbb{N}} G_\ell$. We will show that X' is residual by showing that each G_ℓ is open and dense in X . Since G_ℓ is open by definition, we have only to show that G_ℓ is ε -dense in X for any $\varepsilon > 0$.

Fix $\ell \in \mathbb{N}$ and $\varepsilon > 0$. Let $N \in \mathbb{N}$ be as guaranteed by Proposition 5.1. By Szemerédi's theorem [Sze] and the argument in the proof of Theorem 2.4, there exists $L \in \mathbb{N}$ such that any subset of any geometric progression of length L of relative density at least η contains a geometric progression of length ℓ . Let $F \subseteq S_N$ be a geometric progression of length L . Let $X_\varepsilon \subseteq X$ be as guaranteed by Proposition 5.1. To show that G_ℓ is ε -dense, it suffices now to show that $X_\varepsilon \subseteq G_\ell$.

Let $x \in X_\varepsilon$. By Proposition 5.1, there exists $F' \subseteq F$ with $|F'| > \eta|F|$ and $n \in \mathbb{N}$ such that $nF' \subseteq R(x, U)$. By Szemerédi's theorem, the set F' contains a geometric progression of length ℓ , hence so does nF' . Because $R(x, U)$ contains a geometric progression of length ℓ , the point x belongs to G_ℓ . It follows that $X_\varepsilon \subseteq G_\ell$, as was to be shown. \square

6. RESULTS ON TOTALLY MINIMAL AND DISTAL SYSTEMS

To prove Theorems 1.2 and 1.3, we need the following strengthening of Proposition 5.1 using total visibility; recall Definition 3.3.

Proposition 6.1. *Let (X, T) be an invertible, minimal system and $V \subseteq U \subseteq X$ be open, non-empty sets with $\overline{V} \subseteq U$. If V is totally visible, then there exists $\eta > 0$ and a residual set $X' \subseteq X$ such that for all $x \in X'$ and all finite sets $F \subseteq \mathbb{N}$, there exists $F' \subseteq F$ with $|F'| > \eta|F|$ and an additively invariant mean λ on \mathbb{N} such that for all $f \in F'$, $\lambda(R(x, U)/f) > \eta$.*

Proof. Let μ be a T -invariant probability measure on X for which V is totally visible. Let $\eta > 0$ be a third of the infimum from Definition 3.3. By taking a countable intersection of residual sets, it suffices to show: for all finite sets $F \subseteq \mathbb{N}$,

there exists a residual set $X' \subseteq X$ such that for all $x \in X'$, there exists $F' \subseteq F$ with $|F'| > \eta|F|$ and an additively invariant mean λ on \mathbb{N} such that for all $f \in F'$, $\lambda(R(x, U)/f) > \eta$.

Let $F = \{m_1, \dots, m_\ell\} \subseteq \mathbb{N}$, and put $\vec{m} = (m_1, \dots, m_\ell) \in \mathbb{N}^\ell$ and $M = \text{LCM}(\vec{m})$. Let $X' \subseteq X$ be the set of points $x \in X$ such that $\Delta(x)$ is a point of continuity of the map $\bar{o}_{T^{\vec{m}}} : X^\Delta \rightarrow \mathcal{F}(X^\Delta)$. By Proposition 4.4, X' is a residual subset of X . By the same proposition, for $j \in \{0, \dots, d_M - 1\}$, the set $X'_{M,j} := X' \cap X_{M,j}$ is the set of points $x \in X_{M,j}$ for which $\Delta(x)$ is a point of continuity of the map $\bar{o}_{T^{\vec{m}}} : X_{M,j}^\Delta \rightarrow \mathcal{F}(X_{M,j}^\Delta)$, and $X'_{M,j}$ is residual in $X_{M,j}$.

Fix $j \in \{0, \dots, d_M - 1\}$. The measure $\mu_{M,j}$ is supported on $X_{M,j}$ and, because V is totally visible by μ , satisfies: for all $i \in \{1, \dots, \ell\}$,

$$(13) \quad \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} T^{m_i k} \mu_{M,j}(\bar{V}) = \liminf_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^{N-1} \mu_{M,j+m_i k}(\bar{V}) > 2\eta.$$

Let $\mu_{M,j}^\Delta$ be any weak-* limit point of the set $\{N^{-1} \sum_{n=0}^{N-1} (T^{\vec{m}})^n \Delta(\mu_{M,j}) \mid N \in \mathbb{N}\}$. It follows that $\mu_{M,j}^\Delta$ is a $T^{\vec{m}}$ -invariant probability measure on $X_{M,j}^\Delta$, and (13) implies that for all $i \in \{1, \dots, \ell\}$, $\pi_i \mu_{M,j}^\Delta(\bar{V}) > 2\eta$, where $\pi_i : X^\Delta \rightarrow X$ is the projection onto the i^{th} coordinate.

At this point, we repeat verbatim Paragraphs 3 through 5 of the proof of Proposition 5.1 with $X_{M,j}^\Delta$ in place of X^Δ , $\mu_{M,j}^\Delta$ in place of μ^Δ , and with η in place of $\sqrt{\eta}$. We get $I \subseteq \{1, \dots, \ell\}$ with $|I| > \eta\ell$, $\vec{w} \in X_{M,j}^\Delta$, and a $T^{\vec{m}}$ -invariant probability measure⁴ $\mu_{\vec{w}}^\Delta$ supported on $\bar{o}_{T^{\vec{m}}}(\vec{w})$ with the property that for all $i \in I$, $\pi_i \mu_{\vec{w}}^\Delta(\bar{V}) > \eta$.

Let $x \in X'_{M,j}$, and write $\vec{x} := \Delta(x)$. By Theorem 4.2, the system $(X_{M,j}^\Delta, T^{\vec{m}}, \Delta(T)^M)$ is minimal, so there exists a sequence $((a_n, b_n))_{n \in \mathbb{N}} \subseteq \mathbb{N}^2$ for which $(T^{\vec{m}})^{a_n} \Delta(T)^{b_n M} \vec{w} \rightarrow \vec{x}$ as $n \rightarrow \infty$.

Let ν_x^Δ be a weak-* limit point of the set $\{\Delta(T)^{b_n M} \mu_{\vec{w}}^\Delta \mid n \in \mathbb{N}\}$; by passing to a subsequence, we may assume without loss of generality that $\Delta(T)^{b_n M} \mu_{\vec{w}}^\Delta \rightarrow \nu_x^\Delta$ as $n \rightarrow \infty$. By repeating verbatim Paragraphs 8 and 9 in the proof of Proposition 5.1, we show that ν_x^Δ is a $T^{\vec{m}}$ -invariant probability measure supported on $\bar{o}_{T^{\vec{m}}}(\Delta(x))$ such that for all $i \in I$, $\pi_i \nu_x^\Delta(\bar{V}) > \eta$. An application of Lemma 2.11 just as in Paragraph 10 in the proof of Proposition 5.1 gives the existence of an additively invariant mean λ on \mathbb{N} such that for all $f \in F' := \{m_i \mid i \in I\}$, $\lambda(R(x, U)/f) > \eta$, as was to be shown. \square

It was explained in Remark 3.4 that all non-empty, open sets in a totally minimal system are totally visible by every T -invariant probability measure. We use this fact to prove Theorem 1.2.

Proof of Theorem 1.2. By the same initial argument in the proof of Theorem 1.1, it suffices to prove this theorem in the case that (X, T) is invertible.

By taking a countable basis of open sets and a countable intersection of residual sets, the residual set X' is allowed to depend on the set U . Let $U \subseteq X$ be open, non-empty, and let $V \subseteq U$ be open, non-empty such that $\bar{V} \subseteq U$. Since (X, T) is totally minimal, the set V is totally visible. Let $\eta > 0$ and $X' \subseteq X$ be as

⁴Perhaps a more fitting notation for this measure would be $\mu_{M,j,\vec{w}}^\Delta$, which we avoid for notational simplicity.

guaranteed by Proposition 6.1. We will show that for all $x \in X'$, the set $R(x, U)$ satisfies $d_{\times}^*(R(x, U)) \geq \eta^2$.

Let $x \in X'$ and $F \subseteq \mathbb{N}$ be finite. Let $F' \subseteq F$ and λ be as guaranteed by Proposition 6.1: $|F'| > \eta|F|$, and for all $f \in F'$, $\lambda(R(x, U)/f) > \eta$. By Lemma 2.1, there exists $F'' \subseteq F'$ with $|F''| > \eta|F'| > \eta^2|F|$ and $n \in \mathbb{N}$ such that $nF'' \subseteq R(x, U)$. Since $F \subseteq \mathbb{N}$ was arbitrary, this shows $d_{\times}^*(R(x, U)) \geq \eta^2$. \square

The following lemma gives a sufficient condition on a set $A \subseteq \mathbb{N}$ for all of its translates to have positive multiplicative density in a coset of a multiplicative semi-group, and it will allow us to prove Theorems 1.3 and 1.4.

Lemma 6.2. *Let $A \subseteq \mathbb{N}$, and suppose that there exists $\eta > 0$ and $N \in \mathbb{N}$ for which the following holds:*

$$(14) \quad \begin{aligned} &\text{for all } F \subseteq \mathbb{N} \text{ and } a \in \mathbb{Z}, \text{ there exists } F' \subseteq F \text{ with } |F'| > \eta|F| \\ &\text{and a translation invariant mean } \lambda \text{ on } \mathbb{N} \text{ such that} \end{aligned}$$

$$\text{for all } f \in F', \lambda\left(\frac{A - aN}{Nf}\right) > \eta.$$

Then, for all $t \in \mathbb{Z}$,

$$d_{(N,t)S_{N/(N,t)}}^*(A + t) \geq \eta^2(N, t)/N.$$

Proof. Let $t \in \mathbb{Z}$, and put $K = N/(N, t)$. We will show that $d_{S_K}^*((A + t)/(N, t)) \geq \eta^2/K$.

Let $F \subseteq S_K$ be finite, and let $F' \subseteq F$ with $|F'| \geq |F|/K$ be such that all elements of F' are congruent modulo K to some $f_0 \in S_K$. Denote by $\Pi F'$ the product of the elements of F' . Since $(\Pi F', K) = 1$, there exists $a \in \mathbb{Z}$ and $b \in \mathbb{N}$ with $(b, K) = (t/(N, t), K) = 1$ so that

$$b\Pi F' - aK = \frac{t}{(N, t)}.$$

Put $c = bf_0^{|F'|-1}$ and note that $(c, K) = 1$. Let $f \in F'$. Since $b\Pi F'/f \equiv c \pmod{K}$, $b\Pi F' \equiv cf \pmod{fK}$. Therefore,

$$(15) \quad \begin{aligned} cf - aK &\equiv \frac{t}{(N, t)} \pmod{fK}, \text{ whereby} \\ c(N, t)f - aN &\equiv t \pmod{fN}. \end{aligned}$$

Summarizing, we have found $a \in \mathbb{Z}$ and $c \in \mathbb{N}$ with $(c, K) = 1$ so that for all $f \in F'$, the congruence in (15) holds.

By the assumptions in (14), there exists $F'' \subseteq F'$ with $|F''| > \eta|F'|$ and a translation invariant mean λ on \mathbb{N} such that for all $f \in F''$, $\lambda((A - aN)/(Nf)) > \eta$. For all $f \in F''$, $-aN \equiv t - c(N, t)f \pmod{Nf}$, so by the translation invariance of λ ,

$$\lambda\left(\frac{A + t - c(N, t)f}{Nf}\right) = \lambda\left(\frac{A - aN}{Nf}\right) > \eta.$$

By Lemma 2.1, there exists $F''' \subseteq F''$ with $|F'''| > \eta|F''|$ such that $\bigcap_{f \in F'''}(A + t - c(N, t)f)/(Nf)$ is non-empty; let n be an element of this set. Now $Nn + c(N, t) \in \mathbb{N}$ is such that $(Nn + c(N, t))F''' \subseteq A + t$, meaning that

$$(Kn + c)F''' \subseteq \frac{A + t}{(N, t)}.$$

Since $(Kn + c, K) = (c, K) = 1$, we have shown that $Kn + c \in S_K$ satisfies $|(Kn + c)F \cap (A + t)/(N, t)| \geq |F'''| > \eta^2|F|/K$. Since $F \subseteq S_K$ was arbitrary, this shows $d_{S_K}^*((A + t)/(N, t)) \geq \eta^2/K$. \square

We are now able to prove the following theorem, a strengthening of Theorem 1.3.

Theorem 6.3. *Let (X, T) be a minimal distal system. There exists a residual set $X' \subseteq X$ such that for all non-empty, open $U \subseteq X$, there exists $N \in \mathbb{N}$ and $\eta > 0$ such that for all $x \in X'$, there exists $i \in \mathbb{N}$ such that for all $t \in \mathbb{Z}$,*

$$d_{(N,t)S_{N/(N,t)}}^*(R(x, U) + i + t) \geq \eta(N, t)/N.$$

In particular, putting $t = -i$, we see that the set $R(x, U)$ has positive multiplicative density in a coset of a multiplicative subsemigroup of \mathbb{N} .

Proof. Because (X, T) is distal, it is invertible. By taking a countable basis of open sets and a countable intersection of residual sets, the residual set X' is allowed to depend on the set U . Let $U \subseteq X$ be open, non-empty, and let $V \subseteq U$ be open, non-empty with $\overline{V} \subseteq U$. By Lemma 3.5, there exists $N \in \mathbb{N}$ and $j \in \{0, \dots, d_N - 1\}$ such that the set $V_{N,j}$ is totally visible in the system $(X_{N,j}, T^N)$.

Now $(X_{N,j}, T^N)$ is an invertible, minimal system, $\overline{V}_{N,j} \subseteq U_{N,j}$, and $V_{N,j}$ is totally visible. Let $\sigma > 0$ be the “ η ” as guaranteed by Proposition 6.1, and let $X'_{N,j} \subseteq X_{N,j}$ be as guaranteed by the same proposition. Put $\eta = \sigma^2$. Since $X'_{N,j}$ is residual, so is $\bigcap_{n \in \mathbb{Z}} T^{Nn} X'_{N,j}$; thus, replacing the former set with the latter, we may assume that $X'_{N,j}$ is T^N -invariant.

We will verify next that for every $x \in X'_{N,j}$, the set $R(x, U)$ satisfies the conditions in (14) in Lemma 6.2 with σ as “ η ” and N as it is. Let $x \in X'_{N,j}$ and put $A = R(x, U)$. Let $F \subseteq \mathbb{N}$ and $a \in \mathbb{Z}$. Because $X'_{N,j}$ is T^N -invariant, $T^{aN}x \in X'_{N,j}$. By Proposition 6.1, there exists $F' \subseteq F$ with $|F'| > \sigma|F|$ and an additively invariant mean λ on \mathbb{N} such that for all $f \in F'$,

$$\lambda\left(\frac{A - aN}{Nf}\right) = \lambda\left(\frac{R_{T^N}(T^{aN}x, U)}{f}\right) > \sigma.$$

This shows that the conditions in Lemma 6.2 are satisfied.

Put $X' = \bigcup_{i=0}^{d_N-1} T^i X'_{N,j}$. Since T is a homeomorphism, the set X' is residual. We will show that every $x \in X'$ satisfies the conclusions of the theorem.

Let $x \in X'$. By the definition of X' , there exists $i \in \mathbb{N}$ so that $T^{-i}x \in X'_{N,j}$. Since $R(T^{-i}x, U) = R(x, U) + i$ satisfies the conditions in Lemma 6.2, for all $t \in \mathbb{Z}$,

$$d_{(N,t)S_{N/(N,t)}}^*(R(x, U) + i + t) \geq \sigma^2(N, t)/N = \eta(N, t)/N,$$

as was to be shown. \square

7. RESULTS ON IP_r^* SETS AND NILSYSTEMS

Not every syndetic subset of \mathbb{N} arises as the set of returns $R(x, U)$ in a dynamical system (see Examples 8.2 and 8.4 below), so Theorem 1.1 lends only some evidence in favor of the conjecture that all syndetic subsets of \mathbb{N} contain arbitrarily long geometric progressions. In this section, we show that a subclass of syndetic sets, translates of IP_r^* sets, do have positive multiplicative density in cosets of multiplicative subsemigroups of \mathbb{N} and, hence, are GP-rich.

Definition 7.1. A subset of \mathbb{N} is called IP_r , $r \in \mathbb{N}$, if it contains a *finite sums set with r generators*, a set of the form

$$(16) \quad \text{FS}(x_1, \dots, x_r) := \left\{ \sum_{i \in I} x_i \mid \emptyset \neq I \subseteq \{1, \dots, r\} \right\}, \quad x_1, \dots, x_r \in \mathbb{N}.$$

A set which is IP_r for all $r \in \mathbb{N}$ is called IP_0 . A subset of \mathbb{N} is called IP_r^* if it has non-empty intersection with every IP_r set in \mathbb{N} , and it is called IP_0^* if it has non-empty intersection with every IP_0 set in \mathbb{N} (equivalently, if it is IP_r^* for some $r \in \mathbb{N}$). The *rank* of an IP_0^* set is the minimal $r \in \mathbb{N}$ for which it is an IP_r^* set.

As a warm-up to the proof of Theorem 1.4, we will show that every IP_r^* set A is syndetic. Let $F \subseteq \mathbb{N} \setminus A$ be a maximal finite sums set, and put $F_0 = F \cup \{0\}$. For every $m \in \mathbb{N}$, $F \cup (F_0 + m)$ is a finite sums set that, by the maximality of F , has non-empty intersection with A . Since $A \cap F = \emptyset$, it must be that $A \cap (F_0 + m) \neq \emptyset$, meaning $m \in A - F_0$. Since m was arbitrary, $A - F_0 = \mathbb{N}$, meaning A is syndetic: the set $F_0 + 1 \subseteq \mathbb{N}$ is such that $\bigcup_{f \in F_0+1} (A - f) = \mathbb{N}$.

In fact, we get a quantitative measure on the syndeticity of the set A . Since A is IP_r^* , F is IP_s for some $s \leq r - 1$, meaning $|F_0 + 1| \leq 2^s$. It follows by additivity that for any additively invariant mean λ on \mathbb{N} , $\lambda(A) \geq 2^{-s}$. This is the basis for applying Lemma 6.2 in the proof of the following theorem, from which Theorem 1.4 immediately follows.

Theorem 7.2. *Let $A \subseteq \mathbb{N}$ be an IP_r^* set. There exists an $N \in \mathbb{N}$ such that for all $t \in \mathbb{Z}$,*

$$(17) \quad d_{(N,t)S_{N/(N,t)}}^*(A+t) \geq \frac{(N,t)}{2^{2r+2N}}.$$

Proof. It is quick to check that for all $n \in \mathbb{N}$, $\text{rank}(A/n) \leq \text{rank}(A) \leq r$. It follows that there exists $N \in \mathbb{N}$ for which $\text{rank}(A/N) = \min_{n \in \mathbb{N}} \text{rank}(A/n)$. Let $s = \text{rank}(A/N) \leq r$ be this minimal rank.

We will show that A satisfies the conditions in Lemma 6.2 with N as it is and $\eta = 2^{-(s+1)}$. It suffices to show the following: *for all $f \in \mathbb{N}$ and $a \in \mathbb{Z}$, at most 2^s -many translates of the set $(A - aN)/(Nf) = (A/N - a)/f$ are sufficient to cover all but finitely many elements of \mathbb{N} .* Indeed, this ensures that for every translation invariant mean λ on \mathbb{N} , $\lambda((A - aN)/(Nf)) \geq 2^{-s} > \eta$.

Let $f \in \mathbb{N}$ and $a \in \mathbb{Z}$. Since $\text{rank}(A/fN) = s$, there exists an IP_{s-1} set $G' \subseteq \mathbb{N} \setminus (A/fN)$. (If $s = 1$, then take $G' = \emptyset$.) Set $G := fG' \cup \{0\}$, and note that $|G| \leq 2^s$. We claim that $A/N - G = \mathbb{N}$. Let $m \in \mathbb{N}$. Since $(fG') \cup (G+m)$ is an IP_s set and $(fG') \cap (A/N) = \emptyset$ and $\text{rank}(A/N) = s$, we see that $(A/N) \cap (G+m) \neq \emptyset$, implying that $m \in A/N - G$.

Since $A/N - G = \mathbb{N}$, the set $\mathbb{N} \setminus (A/N - a - G)$ is finite. Dividing by f , we see that $\mathbb{N} \setminus ((A/N - a)/f - (G' \cup \{0\}))$ is co-finite, as was to be shown. \square

As a corollary to Theorem 7.2 and Theorem 2.5, we see that translates of IP_r^* sets contain arbitrarily long geo-arithmetic configurations and so, in particular, are GP-rich.

We now derive three consequences of Theorem 7.2 based on the connection between nilsystems and IP_0^* sets discussed in Section 1.1. A *nilsystem* is a topological dynamical system (X, T) where X is a compact homogeneous space of a nilpotent Lie group G and T is a translation of X by an element of G . The key fact in each

of these consequences follows from [BL, Theorem 0.2]: *in a nilsystem (X, T) , for all non-empty, open $U \subseteq X$ and all $x \in U$, the set $R(x, U)$ is IP_0^* .*⁵

First, we show that sets of returns in minimal nilsystems have positive multiplicative density.

Corollary 7.3. *Let (X, T) be a minimal nilsystem. For all $x \in X$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ has positive multiplicative density in a coset of a multiplicative subsemigroup of \mathbb{N} .*

Proof. Because (X, T) is minimal, there exists $n \in \mathbb{N}$ such that $T^n x \in U$. By [BL, Theorem 0.2], the set $R(T^n x, U)$ is IP_0^* . It follows from Theorem 7.2 that translates of $R(T^n x, U)$ have positive multiplicative density in a coset of a multiplicative subsemigroup of \mathbb{N} . To finish, note that $R(x, U) \supseteq R(T^n x, U) + n$, a translate of $R(T^n x, U)$. \square

A natural question is whether or not our main results can be enhanced by improving “positive upper Banach density” to “multiplicatively piecewise syndetic.” A set $A \subseteq \mathbb{N}$ is *multiplicatively syndetic* if there exists a finite set $F \subseteq \mathbb{N}$ such that $\bigcup_{f \in F} A/f = \mathbb{N}$. A set $C \subseteq \mathbb{N}$ is *multiplicatively piecewise syndetic* if there exists a multiplicatively syndetic set $A \subseteq \mathbb{N}$ and a set $B \subseteq \mathbb{N}$ with $d_\times^*(B) = 1$ such that $C = A \cap B$. Multiplicatively piecewise syndetic sets have positive multiplicative upper Banach density, but, by [BCRZK, Theorem 6.4], there exist subsets of \mathbb{N} of multiplicative density arbitrarily close to 1 that are not multiplicatively piecewise syndetic.

We will argue now that the conclusion of Theorem 1.2 cannot be improved to show that the set $R(x, U)$ is, in general, multiplicatively piecewise syndetic. Suppose (X, T) is a totally minimal nilsystem and U and V are non-empty, disjoint open sets. Let $x \in U$ and put $A = R(x, U)$ and $B = R(x, V)$. Since $U \cap V = \emptyset$, $A \cap B = \emptyset$. By [BL, Theorem 0.2], the set A is an IP_0^* set, and [BG, Corollary 7.3] gives that A has non-empty intersection with all multiplicatively piecewise syndetic subsets of \mathbb{N} . Since $A \cap B = \emptyset$, it follows that the set B is not multiplicatively piecewise syndetic.

Third, we prove Corollary 1.5 from the introduction, an application of Theorem 7.2 to finding geo-arithmetic configurations in sets arising from polynomial Diophantine approximation.

Proof of Corollary 1.5. By writing the set A as a set of return times of a point to an open set in a minimal nilsystem, it is shown in [BG, Theorem 6.14] that A is a translated IP_0^* set. It follows by Theorem 7.2 that there exists $K \in \mathbb{N}$ such that A has positive multiplicative upper Banach density in a coset of the multiplicative subsemigroup S_K . The stated geo-arithmetic configurations can be found in A by using Theorem 2.5. \square

8. SYNDETIC SETS NOT ARISING FROM DYNAMICS

As mentioned in Section 1.2, the fact that geometric progressions and multiplicative density are not translation invariant prevents us from being able to deduce results on arbitrary syndetic subsets of \mathbb{N} from our dynamical ones. Still, one might

⁵Strictly speaking, [BL, Theorem 0.2] concerns invertible systems and IP_0^* subsets of \mathbb{Z} . This theorem applies to our setting by noting that nilsystems are invertible and that if a set A is IP_r^* in \mathbb{Z} , then $A \cap \mathbb{N}$ is IP_r^* in \mathbb{N} .

hope that an arbitrary syndetic set or IP_r^* set takes the form $R(x, U)$, or at least contains a set of the form $R(x, U)$, where x and U are a point and a non-empty, open set in a minimal system or nilsystem. We show in this section that this is not the case.

Lemma 8.1. *Let (X, T) be a minimal system, $x \in X$, and $U \subseteq X$ open, non-empty. There exists $N \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ and all $n \in S_N$, the set $(R(x, U) - t)/n$ is syndetic.*

Proof. Let $N \in \mathbb{N}$ be from Lemma 3.1, and let $t \in \mathbb{N}$ and $n \in S_N$. Since $\bigcup_{\ell=1}^{\infty} (T^n)^{-\ell} U = X$ and X is compact, for all $y \in X$, the set $R_{T^n}(y, U)$ is syndetic; in particular, the set $R_{T^n}(T^t x, U) = (R(x, U) - t)/n$ is syndetic, as was to be shown. \square

Example 8.2. There exists a syndetic set $A \subseteq \mathbb{N}$ such that for all $t, n \in \mathbb{N}$ with $n \geq 2$, the set $(A - t)/n$ is not syndetic. By Lemma 8.1, it follows that A does not contain a set of the form $R(x, U)$ where x and U are a point and a non-empty, open subset of a minimal dynamical system.

To construct such a set, let $\{T_{t,n}\}_{t,n \in \mathbb{N}, n \geq 2}$ be a family of thick subsets of \mathbb{N} (i.e., subsets containing arbitrarily long intervals) with the property that if $m \in T_{t,n}$ and $m + 1 \in T_{t',n'}$, then $t = t'$ and $n = n'$. Put

$$A = \left(\bigcup_{\substack{t,n \in \mathbb{N} \\ n \geq 2}} (T_{t,n} \setminus (n\mathbb{N} + t)) \right) \cup \left(\mathbb{N} \setminus \bigcup_{\substack{t,n \in \mathbb{N} \\ n \geq 2}} T_{t,n} \right).$$

We claim that A is syndetic; in fact, we will show that $A \cup (A - 1) = \mathbb{N}$. Indeed, let $m \in \mathbb{N}$. If $m \notin \bigcup T_{t,n}$, then $m \in A$. Otherwise, there exists $t, n \in \mathbb{N}$ with $n \geq 2$ such that $m \in T_{t,n}$. If $m + 1 \notin \bigcup T_{t,n}$, then $m \in A - 1$. Otherwise, $m + 1 \in \bigcup T_{t,n}$, which implies that $m + 1 \in T_{t,n}$. Since $n \geq 2$, at least one of m and $m + 1$ is in the set $T_{t,n} \setminus (n\mathbb{N} + t)$, meaning $m \in A \cup (A - 1)$. In any case, we have shown that $m \in A \cup (A - 1)$, which, since $m \in \mathbb{N}$ was arbitrary, implies that $A \cup (A - 1) = \mathbb{N}$. By construction, however, for all $t, n \in \mathbb{N}$ with $n \geq 2$, the set $(A - t)/n$ has empty intersection with the thick set $(T_{t,n} - t)/n$, meaning $(A - t)/n$ is not syndetic.

Lemma 8.3. *Let (X, T) be a minimal nilsystem, $x \in X$, and $U \subseteq X$ open, non-empty. The set $R(x, U)$ is non-empty, and for all $t \in R(x, U)$, the set $R(x, U) - t$ is IP_0^* .*

Proof. Because (X, T) is minimal, the set $R(x, U)$ is non-empty. Let $t \in R(x, U)$. Because $T^t x \in U$, it follows from [BL, Theorem 0.2] that the set $R(x, U) - t = R(T^t x, U)$ is IP_0^* . \square

Example 8.4. There exists an IP_2^* set $A \subseteq \mathbb{N}$ such that for all $t \in \mathbb{Z} \setminus \{0\}$, the set $A - t$ is not IP_0^* . By Lemma 8.3, it follows that A does not contain a set of the form $R(x, U)$ where x and U are a point and a non-empty, open subset of a minimal nilsystem.

To construct such a set, let $(m_n)_{n \in \mathbb{N}} \subseteq \mathbb{Z} \setminus \{0\}$ be a sequence with the property that for all $t \in \mathbb{Z} \setminus \{0\}$, there are infinitely many $n \in \mathbb{N}$ such that $m_n = t$. Choose a sequence $(r_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ that is increasing sufficiently rapidly so that the set $B := \bigcup_{n \in \mathbb{N}} (r_n \{1, \dots, n\} + m_n) \subseteq \mathbb{N}$ is not IP_2 , that is, does not contain a

configuration of the form $\{x, y, x + y\}$. For $t \in \mathbb{Z} \setminus \{0\}$, define

$$B_t := \bigcup_{\substack{n \in \mathbb{N} \\ m_n = t}} (r_n \{1, \dots, n\} + t) \subseteq B,$$

and note that $B_t - t$ is an IP_0 set. Set $A = \mathbb{N} \setminus B$. Since B is not IP_2 , the set A is IP_2^* , and for $t \in \mathbb{Z} \setminus \{0\}$, $(A - t) \cap (B_t - t) = \emptyset$, implying that $A - t$ is not IP_0^* .

9. CONCLUDING REMARKS AND QUESTIONS

We collect here a number of further questions and open problems, beginning with ones of a dynamical nature.

There are two primary avenues for improvement in the main dynamical theorems, Theorems 1.1, 1.2, and 1.3: upgrading the conclusions by saying more about the multiplicative combinatorial structure of return time sets $R(x, U)$, and enlarging the set of points X' about which we can address the sets $R(x, U)$. In the first direction, it is natural to speculate how much the conclusion of Theorem 1.1 can be upgraded.

Question 9.1. *Let (X, T) be a minimal dynamical system. Does there exist a residual set of points $X' \subseteq X$ such that for all $x \in X'$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ has positive multiplicative density in a coset of a multiplicative subsemigroup of \mathbb{N} ?*

The conclusion in this question could be further upgraded to, “the set $R(x, U)$ has multiplicative density 1 in a coset of a multiplicative subsemigroup of \mathbb{N} ?” If true, such a result would lend further evidence toward the stronger conjectures about the multiplicative combinatorial structure of additively syndetic sets outlined below.

In the second direction, it is natural to ask about the nature of return time sets $R(x, U)$ for points x outside of X' , the residual subset of X that appears in each of the main dynamical theorems. A positive answer to the following question would improve Theorem 1.1.

Question 9.2. *Let (X, T) be a minimal dynamical system. Is it true that for all $x \in X$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ contains arbitrarily long geometric progressions?*

There is a positive answer to Questions 9.1 and 9.2 in the case that (X, T) is an irrational rotation of the 1-torus. It can be shown in that case that for all $x \in \mathbb{T}$ and all non-empty, open $U \subseteq \mathbb{T}$, there exist $n, N \in \mathbb{N}$ such that $d_{nS_N}^*(R(x, U)) = 1$.

While we are not able to answer these questions in more generality, we do know that systems in which the return time sets $R(x, U)$ are multiplicatively large for all points $x \in X$ enjoy some rather strong dynamical properties. The following lemma outlines some of the (equivalent) dynamical consequences of assuming that every return times set $R(x, U)$ is *multiplicatively thick* in \mathbb{N} : for all finite $F \subseteq \mathbb{N}$, there exists $m \in \mathbb{N}$ so that $mF \subseteq R(x, U)$.

Lemma 9.3. *Let (X, T) be a dynamical system, and for $n \in \mathbb{N}$, let $\Delta_n = \Delta_n(X) \subseteq X^n$ be the diagonal $\{(x, \dots, x) \in X^n \mid x \in X\}$. The following are equivalent:*

- (1) *for all $x \in X$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ is multiplicatively thick in \mathbb{N} ;*
- (2) *for all $x \in X$ and all $n \in \mathbb{N}$, the $T \times T^2 \times \dots \times T^n$ -orbit closure of (x, \dots, x) contains the diagonal Δ_n ;*

(3) for all non-empty, open $U \subseteq X$ and all $n \in \mathbb{N}$,

$$\bigcup_{m=1}^{\infty} (T \times T^2 \times \cdots \times T^n)^{-m} (U \times \cdots \times U) \supseteq \Delta_n;$$

(4) for all non-empty, open $U \subseteq X$ and all $n \in \mathbb{N}$,

$$\bigcup_{m=1}^{\infty} \bigcap_{i=1}^n T^{-mi} U = X;$$

(5) for all $x \in X$ and all non-empty, open $U \subseteq X$, the set $R(x, U)$ satisfies: for all $n \in \mathbb{N}$, there exists a finite $F \subseteq \mathbb{N}$ such that for all $\ell \in \mathbb{N}_0$, there exists $m \in F$ such that $\ell + m\{1, \dots, n\} \subseteq R(x, U)$.

Proof. We will show that each condition implies the one following it; that condition (5) implies condition (1) is immediate by taking $\ell = 0$.

(1) implies (2): Let $x \in X$ and $n \in \mathbb{N}$. Any non-empty, open subset V of Δ_n contains a set of the form $(U \times \cdots \times U) \cap \Delta_n$, where $U \subseteq X$ is non-empty, open. By (1), there exists $m \in \mathbb{N}$ such that $m\{1, \dots, n\} \subseteq R(x, U)$. This means that $(T \times T^2 \times \cdots \times T^n)^m(x, \dots, x) \in U \times \cdots \times U$. Since V was arbitrary, this shows that the $T \times T^2 \times \cdots \times T^n$ -orbit closure of (x, \dots, x) contains the diagonal Δ_n .

(2) implies (3): Let $U \subseteq X$ be non-empty, open and $n \in \mathbb{N}$. Let $(x, \dots, x) \in \Delta_n$. By (2), there exists $m \in \mathbb{N}$ such that $(T \times T^2 \times \cdots \times T^n)^m(x, \dots, x) \in U \times \cdots \times U$. This implies that $(x, \dots, x) \in (T \times T^2 \times \cdots \times T^n)^{-m} (U \times \cdots \times U)$.

(3) implies (4): Let $U \subseteq X$ be non-empty, open and $n \in \mathbb{N}$. Let $x \in X$. By (3), there exists $m \in \mathbb{N}$ such that $(x, \dots, x) \in (T \times T^2 \times \cdots \times T^n)^{-m} (U \times \cdots \times U)$, meaning that $x \in \bigcap_{i=1}^n T^{-mi} U$.

(4) implies (5): Let $x \in X$ and $U \subseteq X$ be non-empty, open. Let $n \in \mathbb{N}$. By (4) and the compactness of X , there exists a finite $F \subseteq \mathbb{N}$ such that

$$\bigcup_{m \in F} \bigcap_{i=1}^n T^{-mi} U = X.$$

Let $\ell \in \mathbb{N}_0$. There exists $m \in F$ such that $T^\ell x \in \bigcap_{i=1}^n T^{-mi} U$, meaning that $\ell + m\{1, \dots, n\} \subseteq R(x, U)$. \square

The $n = 1$ case of condition (4) in Lemma 9.3 is equivalent to the minimality of (X, T) . Condition (5) is easily seen to imply that $R(x, U)$ is both additively syndetic (the gap size is bounded by $\max F$) and multiplicatively thick. This is to be expected: as soon as the set $R(x, U)$ is non-empty for all $x \in X$ and all non-empty, open $U \subseteq X$, the system (X, T) must be minimal and hence the sets $R(x, U)$ must be additively syndetic.

We proceed now with some open questions of a combinatorial nature related to the main motivating question, Question 1.6. The most basic open combinatorial question is whether or not syndetic sets contain a square ratio.

Question 9.4. *Do all additively syndetic subsets of \mathbb{N} contain a configuration of the form $\{x, xy^2\}$ for $x, y \in \mathbb{N}$?*

Going beyond square ratios and geometric progressions, the results in Theorems 1.2 and 1.3 suggest that syndetic subsets of \mathbb{N} may have positive multiplicative density in a coset of some multiplicative subsemigroup. In fact, the improvement

of Theorem 1.3 in Theorem 6.3 suggests the possibility that finitely many subsemigroups suffice to capture the multiplicative density of a syndetic set and all of its translates.

Question 9.5. *Let $A \subseteq \mathbb{N}$ be additively syndetic.*

- (1) *Do there exist $n, N \in \mathbb{N}$ such that $d_{nS_N}^*(A) > 0$?*
- (2) *Do there exist $i, N \in \mathbb{N}$ such that for all $t \in \mathbb{Z}$, $d_{(t,N)S_{N/(N,t)}}^*(A+i+t) > 0$?*

We have not even been able to rule out the possibility that syndetic sets have full multiplicative density in a coset of some non-trivial multiplicative subsemigroup. A positive answer to the following question would yield a positive answer not only to Question 1.6, but to Questions 9.1, 9.2, 9.4, and 9.5 (1).

Question 9.6. *Let $A \subseteq \mathbb{N}$ be additively syndetic. Do there exist $n, N \in \mathbb{N}$ such that $d_{nS_N}^*(A) = 1$?*

Being unable to answer Question 9.6 for arbitrary syndetic sets, it makes sense to narrow the scope by asking the same question for combinatorially defined subclasses of syndetic sets.

Question 9.7. *Let $A \subseteq \mathbb{N}$ be additively IP_0^* . Is it true that for all $t \in \mathbb{Z}$, there exist $n, N \in \mathbb{N}$ such that $d_{nS_N}^*(A+t) = 1$?*

It is a consequence of [BG, Corollary 7.3] that the answer to Question 9.7 is “yes” when $t = 0$ with $n = N = 1$. Still, it is entirely possible that some or all of the questions posed here have a negative answer in general.

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