Interpolation sets for dynamical systems

By Andreas Koutsogiannis, Anh N. Le, Joel Moreira,
Ronnie Pavlov and Florian K. Richter

Abstract

Originating in harmonic analysis, interpolation sets were first studied in dynamics by Glasner and Weiss in the 1980s. A set $S \subset \mathbb{N}$ is an interpolation set for a class of topological dynamical systems $\mathcal{C}$ if any bounded sequence on S can be extended to a sequence that arises from a system in $\mathcal{C}$. In this paper, we provide combinatorial characterizations of interpolation sets for:

- (totally) minimal systems;
- topologically (weak) mixing systems;
- strictly ergodic systems; and
- zero entropy systems.

Additionally, we prove some results on a slightly different notion, called weak interpolation sets, for several classes of systems. We also answer a question of Host, Kra, and Maass concerning the connection between sets of pointwise recurrence for distal systems and IP-sets.

Contents

1. Introduction

Several studies have explored the notion of interpolation/interpolating sets for classes of sequences arising from dynamical systems [11, 12, 24, 25, 36]. The first such definition is due to Glasner [10]: letting $\mathcal{A}$ be a subset of $\ell^\infty(\mathbb{N})$ – the space of all bounded complex-valued functions on $\mathbb{N}$ – a set $S \subset \mathbb{N}$ is called an *interpolation set for* $\mathcal{A}$ if every bounded function $f \colon S \to \mathbb{C}$ can be extended to a function in $\mathcal{A}$. The collection of all interpolation sets for $\mathcal{A}$ is denoted by $\mathcal{I}_\mathcal{A}$.

A *topological dynamical system* is a pair (X, T) where X is compact metric space and $T \colon X \to X$ is a continuous map. The system (X, T) is *minimal* if for every $x \in X$, the (forward) orbit $o_T(x) := \{T^n x : n \in \mathbb{N}_0\}$ is dense in X, where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. Let $\mathcal{M} \subset \ell^\infty(\mathbb{N})$ denote the class of bounded sequences x for which the closure of the orbit of x under the left shift map σ forms a minimal dynamical system and let $\mathcal{U}$ be the smallest norm-closed, translation-invariant subalgebra of $\ell^\infty(\mathbb{N})$ containing $\mathcal{M}$. It is an old question asked by Furstenberg [7] whether $\mathcal{U} = \ell^\infty(\mathbb{N})$. Glasner and Weiss [12] provided a negative answer to this question (for $\mathbb{Z}$-subshifts rather than $\mathbb{N}$-subshifts) through an investigation into the interpolation sets for $\mathcal{U}$. If $\mathcal{U} = \ell^\infty(\mathbb{N})$, every subset of $\mathbb{N}$ would be an interpolation set for $\mathcal{U}$. However, Glasner and Weiss [12] showed that an interpolation set for $\mathcal{U}$ cannot be piecewise syndetic (see Section 2 for definition). Recently, Glasner, Tsankov, Weiss, and Zucker [11] generalized this result from $\mathbb{Z}$ to arbitrary infinite discrete groups.

The work of Glasner and Weiss mentioned above was motivated partly by a classical notion in harmonic analysis called Sidon sets. Letting $\mathcal{F} \subset \ell^\infty(\mathbb{N})$ be all Fourier transforms of measures on the unit circle, a set $S \subset \mathbb{N}$ is *Sidon* if it is an interpolation set for $\mathcal{F}$. Sidon [33, 34] himself showed that every lacunary set is Sidon.[1] A later result of Drury [3] showed that the union of two Sidon sets is Sidon.

An important special case of Sidon sets arises when one considers $\mathcal{F}_0$ – the set of Fourier transforms of discrete measures on the unit circle. The interpolation sets for $\mathcal{F}_0$ are called I_0-*sets*. I_0-sets were studied extensively starting in the 1960s [16, 17, 23, 30, 31]. Strzelecki [35] showed that lacunary sets are I_0-sets, extending Sidon's results [33, 34]. Ryll-Nardzewski proved that the union of an I_0-set and a finite set is an I_0-set [31]. Using Hartman-Ryll-Nardzewski characterization [17], some unions of lacunary sets such as $\{2^n : n \in \mathbb{N}\} \cup \{2^n + 1 : n \in \mathbb{N}\}$ are I_0-sets. However, in contrast to Sidon sets, the union of two I_0-sets is not necessarily I_0, for example, $\{2^n : n \in \mathbb{N}\} \cup \{2^n + n : n \in \mathbb{N}\}$ [31]. It is not true either that every I_0-set is a finite union of lacunary sets as seen in the following example by Grow [14] and Méla [27]: $\{3^{n^2} + 3^j : n \geqslant 1, (n-1)^2 \leqslant j \leqslant n^2\}$.

The building blocks of previously known examples of I_0-sets are lacunary sets, a feature that makes them extremely sparse. Therefore, it is natural to ask whether there exists an I_0-set that has subexponential[2] or even polynomial growth. Using a dynamical method, the second author [24] gave a negative answer to this question. This result was extended to interpolation sets for nilsequences in [2, 25]. Another connection between interpolation sets and dynamics was

---

[1] A set $\{a_n : n \in \mathbb{N}\} \subset \mathbb{N}$ with $a_1 < a_2 < \ldots$ is *lacunary* if $\inf_{n \in \mathbb{N}} a_{n+1}/a_n > 1$.

[2] $\{a_n : n \in \mathbb{N}\} \subset \mathbb{N}$ with $a_1 < a_2 < \ldots$ is *subexponential* if $\lim_{n \to \infty} (\log a_n)/n = 0$.

discovered by Griesmer [13], who established a link between I_0-sets and Katznelson's question on the equivalence of Bohr recurrence and topological recurrence.

Interpolation sets also connect to questions in number theory. Sarnak's Möbius disjointness conjecture [32], for example, lies at the intersection of multiplicative number theory and dynamics and has received a lot of attention in the last decade (see [5, 22] for surveys on recent developments). The fourth author showed in [28] that every set of Banach density 0 is a weak interpolation set (Definition 1.2) for the class of all finite-valued sequences whose shift closure forms a zero-entropy minimal dynamical system. This implies that Sarnak's conjecture fails if one replaces the sequence of integers with any sequence of zero Banach density. In particular, this disproved a polynomial analogue of Sarnak's conjecture posed in [4], which has previously been addressed independently by Kanigowski, Lemańczyk, and Radziwiłł [20] and Lian, Shi [26].

Given a rich but scattered literature on interpolation sets in dynamics, the primary goal of this work is a systematic study of interpolation sets for dynamical systems. Specifically, we characterize the interpolation sets for sequences arising from following classes of topological dynamical systems: systems with various mixing-type properties, minimal and totally minimal systems, uniquely and strictly ergodic systems, and systems with restricted entropy. We begin our exploration with precise definitions.

Definition 1.1 (Strong interpolation)**.** Let $\mathcal{C}$ be a class of topological dynamical systems. A set $S \subset \mathbb{N}$ is an *interpolation set for $\mathcal{C}$* if for every bounded function $f : S \to \mathbb{C}$ there exists a system $(X, T) \in \mathcal{C}$, a transitive[3] point $x \in X$ and $F \in C(X)$ such that $F(T^n x) = f(n)$ for every $n \in S$.

In other words, S is an interpolation set for $\mathcal{C}$ if and only if $S \in \mathcal{I}_{\mathcal{F}}$ for $\mathcal{F} = \{n \mapsto F(T^n x) : (X, T) \in \mathcal{C}, \ F \in C(X), \ x \in X \text{ transitive}\}$. To differentiate with a weaker notion of interpolation sets we are about to introduce, occasionally we refer to interpolation sets as *strong interpolation sets*.

In some situations, it will be useful to have a version of interpolation sets which works for functions from the set S to a finite set, rather than to $\mathbb{C}$. The size of the finite set in some sense quantifies the "amount of independence" to be exhibited. A similar idea was already considered by Weiss [36, Theorem 8.1], where he proved that if X is a $\{0, 1\}$-subshift of positive entropy, then there exists $S \subset \mathbb{N}$ of positive density such that $X|_S = \{0, 1\}|_S$. In that spirit, we make the following definition.

Definition 1.2 (Weak interpolation)**.** Let $\mathcal{C}$ be a class of topological dynamical systems and let $k \in \mathbb{N}$. A set $S \subset \mathbb{N}$ is a *weak interpolation set of order k for $\mathcal{C}$* if for every bounded $f : S \to \{0, \ldots, k-1\}$ there exists a system $(X, T) \in \mathcal{C}$, a transitive point $x \in X$ and $F \in C(X)$ such that $F(T^n x) = f(n)$ for every $n \in S$. We say that S is a *weak interpolation set of all orders for $\mathcal{C}$* if it is a weak interpolation set of order k for $\mathcal{C}$ for all $k \in \mathbb{N}$.

There are some obvious relations between these notions of interpolation sets. It is immediate from the definition that for any $k' \leqslant k$, every weak interpolation set of order k is a weak

---

[3]A point $x \in X$ is *transitive* if its orbit, $o_T(x) = \{T^n x : n \in \mathbb{N}_0\}$, is dense in X.

interpolation set of order k'. Similarly, a strong interpolation set is a weak interpolation set of all orders.

For many classes of systems, the differences between weak interpolation sets of different orders are genuine. For example, since there exists a minimal and uniquely ergodic $\{0,1\}$-subshift of positive entropy [15], Weiss's result [36, Theorem 8.1] implies that there is a positive density set S which is weak interpolation of order 2 for the class of uniquely ergodic systems. On the other hand, in Theorem 1.6 below, we prove that every set of positive Banach density (such as S) cannot be a weak interpolation set of all orders for uniquely ergodic systems.
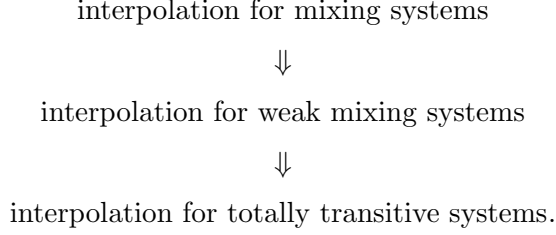
In general, it is also false that weak interpolation of all orders implies strong interpolation. Our Proposition 1.9 shows that for the class of finite entropy systems, all sets (including $\mathbb{N}$ itself) are weak interpolation of all orders, while Proposition 1.8 shows that only sets of zero Banach density are strong interpolation.

That being said, for some special classes of systems, the concepts "weak interpolation of order k" for a specific value of k, "weak interpolation of all orders," and "strong interpolation" are all equivalent. These classes include compact abelian group rotations [17, Theorem 1], nilsystems [25, Theorem 2.2], and minimal systems (Theorem 1.5 below). In general, these concepts are equivalent for the class $\mathcal{C}$ if $\mathcal{C}$ is closed under taking subsystems and countable Cartesian products (see Section 2.3).

The notions of interpolation sets we discussed above also connect to the notions of "null" and "tame" systems, which are important classes with several useful equivalent definitions. *Null systems* are those with zero topological sequential entropy for every subsequence, and *tame systems* are those whose Ellis enveloping semigroup has cardinality at most $2^{\aleph_0}$. For minimal systems, null is strictly stronger than tame, which is strictly stronger than uniquely ergodic with (measurable) discrete spectrum (see [19]). The connection to interpolation sets arises from alternative definitions of these notions in terms of so-called *independence sets* (see [21]). For instance, in [6] it is shown that a system is non-tame if and only if there exist disjoint compact sets K_0, K_1 and an infinite set $S \subset \mathbb{N}$ so that for every $f : S \to \{0,1\}$, there exists $x \in X$ with $T^s x \in K_{f(s)}$ for all $s \in S$. A system is non-null if and only if there exist disjoint compact sets K_0, K_1 and arbitrarily large finite sets S with the above properties. These conditions are not equivalent to ours; for instance, for any non-tame system (X, T), there is an infinite weak interpolation set S of order 2 for the singleton class $\mathcal{C} = \{(X, T)\}$ (by using Urysohn's Lemma with the sets K_0, K_1), but it's not clear whether the converse is true (since in our definition, every function from S to $\{0,1\}$ could correspond to a different $F \in C(X)$).

1.1. Totally transitive, weak mixing, and strong mixing

Three central notions of mixing in topological dynamical systems are (strong) mixing, weak mixing and total transitivity (see Section 2 for the definitions). The following implications are well known: mixing $\Rightarrow$ weak mixing $\Rightarrow$ totally transitive. Therefore, by definition,

interpolation for mixing systems

$$\Downarrow$$

interpolation for weak mixing systems

$$\Downarrow$$

interpolation for totally transitive systems.

Our first result asserts that the converse directions of both implications above also hold, and provides an easy-to-check combinatorial description of interpolation sets for these classes.

Theorem 1.3. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is a weak interpolation set of all orders for totally transitive systems.*
2. *S is an interpolation set for totally transitive systems.*
3. *S is an interpolation set for weak mixing systems.*
4. *S is an interpolation set for mixing systems.*
5. *S is not syndetic, i.e. S has arbitrarily large gaps (see Section 2 for definition).*

We do not know whether weak interpolation of order 2 for totally transitive systems is also equivalent to the condition that S is not syndetic; see Question 8.7.

Remark 1.4. It is natural to wonder why we did not consider interpolation sets for the class of transitive systems. This is because by our definition, every subset of $\mathbb{N}$ is such an interpolation set. Indeed, let $z = (z(n))_{n \in \mathbb{N}_0}$ be an arbitrary bounded sequence (function) and let X be the closure of the orbit of z under the left shift σ. By definition, the system (X, σ) is transitive with z as a transitive point. In addition, $F(\sigma^n z) = z(n)$ for all $n \in \mathbb{N}$ for the continuous function $F : X \to \mathbb{C}$ defined by $F((x(n))_{n \in \mathbb{N}_0}) = x(0)$.

1.2. Minimal and totally minimal systems

Recall that $\mathcal{M}$ denotes the class of bounded sequences whose shift closure forms a minimal dynamical system and $\mathcal{U}$ the closed, shift invariant subalgebra of $\ell^\infty(\mathbb{N})$ generated by $\mathcal{M}$. In [12, Theorem 1], Glasner and Weiss showed that S is an interpolation set for $\mathcal{U}$ if and only if S is not piecewise syndetic.[4] Since $\mathcal{M} \subset \mathcal{U}$, it follows that an interpolation set for $\mathcal{M}$ must be non-piecewise syndetic. However, because the inclusion $\mathcal{M} \subset \mathcal{U}$ is strict, the converse direction is unknown, i.e., is every non-piecewise syndetic set an interpolation set for $\mathcal{M}$? Our next theorem confirms that this is indeed the case. In addition, we show that every non-piecewise syndetic set is an interpolation set for a smaller class, namely, totally minimal systems.

Theorem 1.5. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is a weak interpolation set of order 2 for minimal systems.*
2. *S is a weak interpolation set of all orders for minimal systems.*
3. *S is an interpolation set for minimal systems.*

---

[4]See Section 2 for definition; in [12], non-piecewise syndetic sets are called *small sets*.

4. S is an interpolation set for totally minimal systems.

5. S is not piecewise syndetic.

1.3. Uniquely ergodic and strictly ergodic systems

A topological system (X, T) is *uniquely ergodic* if there is only one Borel measure μ on X that is preserved by T (i.e., $\mu(A) = \mu(T^{-1}A)$ for every Borel set $A \subset X$), and is *strictly ergodic* if it is uniquely ergodic and minimal. The next theorem characterizes interpolation sets for uniquely ergodic systems as well as for strictly ergodic systems.

Theorem 1.6. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is a weak interpolation set of all orders for uniquely ergodic systems.*

2. *S is an interpolation set for uniquely ergodic systems.*

3. *S is an interpolation set for strictly ergodic systems.*

4. *S has zero Banach density, i.e., $d^*(S) = 0$ (see Section 2 for definition).*

As we have mentioned before, by Hahn-Katznelson's [15] and Weiss's results [36, Theorem 8.1], there is a set S of positive density such that S is weak interpolation of order 2 for the class of strictly ergodic systems. This fact and Theorem 1.6 imply two things: First, for the classes of uniquely ergodic systems and strictly ergodic systems, weak interpolation of order 2 and weak interpolation of all orders are not equivalent. Second, it leads to the question: Can a weak interpolation set of order 2 for uniquely ergodic systems be syndetic? At the moment, we do not know the answer to this question (see Question 8.8). However, the answer is negative if we impose an additional condition on the growth of the corresponding complexity function:

Proposition 1.7. *If $S \subset \mathbb{N}$ is syndetic and the word complexity of the sequence 1_S grows subexponentially (i.e. the orbit closure of 1_S under the left shift has zero entropy), then S is not a weak interpolation set of order 2 for uniquely ergodic systems.*

1.4. Systems of bounded entropy

Topological entropy is a measure of the complexity of a dynamical system. Since the definition of entropy is rather technical, we delay it until Section 2. It turns out that classifying interpolation sets for the class of systems whose entropy is bounded by a fixed constant is simple: in the definition of interpolation sets we allow the function to be extended to take infinitely many values, and so interpolation sets for finite entropy systems must have zero Banach density.

Proposition 1.8. *Let $S \subset \mathbb{N}$. The followings are equivalent:*

1. *S is an interpolation set for systems of finite topological entropy.*

2. *S is an interpolation for systems of topological entropy $\leqslant M$ for some fixed $M \geqslant 0$.*

3. *S is an interpolation set for systems of zero topological entropy.*

4. *$d^*(S) = 0$.*

On the other hand, every subset of $\mathbb{N}$ is a weak interpolation set of all orders for finite entropy

6

systems. This is a consequence of the fact that the full shift on a finite alphabet has a finite entropy.

Proposition 1.9. *Every subset $S \subset \mathbb{N}$ is a weak interpolation set of all orders for systems of finite entropy.*

Two results above show that for the class of finite entropy systems, strong interpolation and weak interpolation of all orders are not equivalent.

Due to the aforementioned simple characterizations, when working with finite entropy systems, it is more interesting to consider weak interpolation sets of a fixed order k. In this regard, there is an interesting relation between the entropy bound and the density of the weak interpolation sets. To state the next theorem, we define the entropy function $H : [0,1] \to [0, \log 2]$ as

$$H(\delta) = \begin{cases} -\delta \log \delta - (1 - \delta) \log(1 - \delta), & \text{if } \delta \in (0,1) \\ 0, & \text{if } \delta = 0 \text{ or } 1. \end{cases}$$

Theorem 1.10. *Let $\delta \in [0, 1/2]$. For any $k \in \mathbb{N}$ and $S \subset \mathbb{N}$, we have the following:*
1. *If $d^*(S) \leqslant \delta$, then S is a weak interpolation set of order k for systems of entropy $\leqslant H(\delta) + \delta \log(k - 1)$.*
2. *If $d^*(S) > \delta$, then S is not a weak interpolation set of order k for the class of systems of entropy $\leqslant \delta \log k$.*

Moreover, (2) is sharp in the following sense:
3. *There exists $S \subset \mathbb{N}$ with $d^*(S) = \delta$ such that S is a weak interpolation set of order k for the class of systems of entropy $\leqslant \delta \log k$.*

We do not know whether (1) is sharp, i.e. whether the bound $H(\delta) + \delta \log(k - 1)$ could be improved, and leave this as a question.

Outline of paper. Section 2 contains some background and general discussion. Sections 3-6 concern, respectively, totally transtive/weak mixing/mixing systems, minimal systems, strictly ergodic systems, and systems with bounded entropy. In Section 7 we prove some results about distal systems, including an answer to a question of Host, Kra, and Maass [18] on the connection between sets of pointwise recurrence for distal systems and IP-sets. Finally, Section 8 contains some open questions that naturally arise from our study.

2. Background

2.1. Some families of subsets of integers

Definition 2.1. Let S be a nonempty subset of $\mathbb{N}$.
- S is *syndetic* if the gaps between consecutive elements of S are bounded. Equivalently, S is syndetic if there exists $k \in \mathbb{N}$ such that $S \cup (S - 1) \cup \ldots \cup (S - k) \supset \mathbb{N}$.
- S is *thick* if it contains arbitrary long intervals of the form $\{m, m + 1, \ldots, n\}$.

- S is *piecewise syndetic* if it is the intersection of a syndetic set and a thick set.
- S is *thickly syndetic* if for every $n \in \mathbb{N}$, there exists a syndetic set E such that $\bigcup_{m \in E} \{m, m + 1, \ldots, m + n\} \subset S$. Equivalently, S is thickly syndetic if and only if its complement $\mathbb{N} \backslash S$ is not piecewise syndetic.

  The classes of thickly syndetic and piecewise syndetic sets are dual in the following sense: every thickly syndetic set intersects every piecewise syndetic set and vice versa.

Definition 2.2. The *upper Banach density of* $S \subset \mathbb{N}$ is

$$d^*(S) = \sup_{(F_N)_{N \in \mathbb{N}}} \limsup_{N \to \infty} \frac{|S \cap F_N|}{|F_N|}$$

where the supremum is taken over all sequences of intervals $(F_N)_{N \in \mathbb{N}}$ in $\mathbb{N}$ satisfying $|F_N| \to \infty$ as $N \to \infty$. An equivalent definition of the upper Banach density of S is

$$d^*(S) = \lim_{n \to \infty} \sup_{m \in \mathbb{N}} \frac{|S \cap \{m, \ldots, m + n - 1\}|}{n}.$$

If $d^*(S) = 0$, then $\limsup_{N \to \infty} \frac{|S \cap F_N|}{|F_N|} = 0$ for any sequence $(F_N)_{N \in \mathbb{N}}$ of intervals in $\mathbb{N}$ whose lengths tend to infinity. In this case, we say S has *zero Banach density*.

2.2. Some classes of dynamical systems

Definition 2.3. A *topological dynamical system* (or a *system* for short) is a pair (X, T), where X is a compact metric space and $T : X \to X$ is a continuous map.

A point $x \in X$ is *transitive* if its forward orbit $o_T(x) := \{T^n x : n \in \mathbb{N}_0\}$ is dense in X.

Definition 2.4. A system (X, T) is

- *transitive* if it contains a transitive point,

- *totally transitive* if (X, T^k) is transitive for every $k \in \mathbb{N}$,

- (topological) *weakly mixing* if for any open $U, V \subset X$, the set $\{n \in \mathbb{N} : T^{-n}U \cap V \neq \varnothing\}$ is thick,

- (topological) *mixing* if for any open $U, V \subset X$, the set $\{n \in \mathbb{N} : T^{-n}U \cap V \neq \varnothing\}$ is co-finite,

- *uniquely ergodic* if there exists a unique Borel probability measure μ on X satisfying $\mu(A) = \mu(T^{-1}A)$ for every Borel set $A \subset X$,

- *totally uniquely ergodic* if (X, T^k) is uniquely ergodic for every $k \in \mathbb{N}$.

Remark 2.5. There are several other notions of transitivity in topological dynamics. Among them, the most commonly used is probably the following: A system (X, T) is *set transitive* if for any open U, V, there exists $n \in \mathbb{N}$ such that $U \cap T^{-n}V \neq \varnothing$; and (X, T) is *set totally transitive* if (X, T^k) is set transitive for all $k \in \mathbb{N}$.

When X is a compact metric space, set transitivity implies (point) transitivity – the notion we introduce in Definition 2.4. Likewise, set total transitivity implies (point) total transitivity. The converse implications of both statement also hold if X has no isolated point. (See [1] for a discussion on various notions of transitivity and their relations.)

Definition 2.6. Let (X, T) be a system and let ρ be a metric on X. Let $\varepsilon > 0$. If $c(n, \varepsilon, T)$ denotes the minimum cardinality of covers of X of sets with ρ_n-diameter less than ε, where $\rho_n(x, y) = \max_{0 \leqslant k \leqslant n} \rho(T^k x, T^k y)$, we define as *topological entropy* of the system the (finite) number

$$h(T) = \lim_{\varepsilon \to 0^+} \lim_{n \to \infty} \frac{\log(c(n, \varepsilon, T))}{n}.$$

Definition 2.7. Let $\mathcal{A}$ be a compact set and define $\Omega_{\mathcal{A}} = \mathcal{A}^{\mathbb{N}_0}$ endowed with the product topology. Every point $x \in \Omega_{\mathcal{A}}$ is a sequence $(x(n))_{n \in \mathbb{N}_0}$ with $x(n) \in \mathcal{A}$. Define the *left shift* $\sigma : \Omega_{\mathcal{A}} \to \Omega_{\mathcal{A}}$ by $\sigma((x(n))_{n \in \mathbb{N}_0}) = (x(n+1))_{n \in \mathbb{N}_0}$.

The dynamical system $(\Omega_{\mathcal{A}}, \sigma)$ is mixing and is called *the full shift on* $\mathcal{A}$. A *subshift on* $\mathcal{A}$ is a pair (X, σ) where X is a closed subset of X satisfying $\sigma(X) \subset X$.

2.3. Equivalence of strong interpolation and weak interpolation for some classes of systems

In this section, we show the equivalence of various notions of interpolation sets for certain classes of systems. In the special case that $\mathcal{C}$ is the class of compact abelian group rotations, this result was proved in [17, Theorem 1]. An analogous result for nilsystems was stated in [25, Theorem 2.2] without proof. Our proof here is similar to the one in [17, Theorem 1].

Proposition 2.8. *If $\mathcal{C}$ is a class of dynamical systems satisfying*
(1) If $(X, T) \in \mathcal{C}$ and (Y, T) is a subsystem of (X, T), then $(Y, T) \in \mathcal{C}$ and
(2) If (X, T) and $(Y, R) \in \mathcal{C}$, then $(X \times Y, T \times R) \in \mathcal{C}$,
then weak interpolation of order 2 for $\mathcal{C}$ is equivalent to weak interpolation of all orders for $\mathcal{C}$. If in addition $\mathcal{C}$ satisfies
(2') If $(X_n, T_n) \in \mathcal{C}$ for $n \in \mathbb{N}$, then $(\prod_n X_n, \prod_n T_n) \in \mathcal{C}$,
then weak interpolation of order 2 for $\mathcal{C}$ is equivalent to strong interpolation for $\mathcal{C}$.

Remark 2.9. We note that (1) and (2) hold for the classes of expansive systems and systems of finite entropy, and that (1) and (2') hold for the classes of systems of zero entropy, compact abelian group rotations, inverse limits of nilsystems, and distal systems.

On the other hand, (1) fails for several classes of systems, eg., systems with positive entropy, transitive systems and (weak) mixing system. Condition (2) fails for minimal systems and uniquely ergodic systems.

Proof. Assume that $\mathcal{C}$ satisfies (1) and (2). Since weak interpolation of order k implies weak interpolation of all smaller orders, it suffices to show that weak interpolation of order 2 implies weak interpolation of order 2^m for all m. Suppose that $S \subset \mathbb{N}$ is a weak interpolation set of order 2 for $\mathcal{C}$ and let $f : S \to \{0, 1, \ldots, 2^m - 1\}$ be arbitrary. We can decompose $f = \sum_{k=0}^{m-1} f_k$ with $f_k : S \to \{0, 2^k\}$ by binary expansion. Then, for each k, since S is weak interpolation of order 2, there exists $(X_k, T_k) \in \mathcal{C}$, a continuous function $F_k : X_k \to [0, 1]$ and a point $x_k \in X_k$

such that

$$F_k(T_k^n x_k) = f_k(n) \text{ for all } n \in S.$$

Define the orbit closure Y of the point $x := (x_k)_{0 \leqslant k < m}$ in the product system $\prod_{k=0}^{m-1}(X_k, T_k)$. Denote $T = \prod_{k=1}^{\infty} T_k$. By (1) and (2), since (Y, T) is a subsystem of a finite product of (X_k, T_k), $(Y, T) \in \mathcal{C}$. Finally, define the continuous function $F \in C\left(\prod_{k=1}^{\infty} X_k\right)$ as $F(y_0, \ldots, y_{m-1}) = \sum_{k=0}^{m-1} 2^k F_k(y_k)$ and note that $F(T^n x) = f(n)$ for $n \in S$. This shows that S is weak interpolation of order 2^m for $\mathcal{C}$, and since m was arbitrary, of all orders.

The proof of the second part is similar. Again, we need only prove that if $\mathcal{C}$ satisfies (1) and (2') and $S \subset \mathbb{N}$ is weak interpolation of order 2 for $\mathcal{C}$, then it is interpolation for $\mathcal{C}$. Consider any such S and any bounded function $f : S \to \mathbb{C}$. By the previous part of the proof, we know that S is weak interpolation of all orders. Letting $C := \|f\|_\infty$, we can write $f = C\left(\sum_{k=1}^{\infty} f_k + ig_k\right)$ with $f_k, g_k : S \to \{0, \pm 2^{-k}\}$Then we proceed exactly as above, defining $(X_k, T_k), (Y_k, S_k) \in \mathcal{C}$, $x_k \in X_k$, $y_k \in Y_k$, $F_k \in C(X_k)$, and $G_k \in C(Y_k)$ so that

$$F_k(T_k^n x_k) = f_k(n), G_k(S_k^n y_k) = g_k(n) \text{ for all } n \in S.$$

Now we again take the orbit closure of the sequence $((x_k, y_k))_{k \in \mathbb{N}}$ in the infinite product system, which is in $\mathcal{C}$ by (1) and (2'). The function $F : \prod_{k=1}^{\infty} X_k \to \mathbb{C}$ defined by $F((x_k, y_k))_{k \in \mathbb{N}}) = C\sum_{k=1}^{\infty}(F_k(x_k) + iG_k(y_k))$ is a uniform limit of continuous functions, and therefore continuous. Just as before, $F(T^n x) = f(n)$ for all $n \in S$, verifying that S is an interpolation set for $\mathcal{C}$. $\qquad\square$

3. Totally transitive, weak mixing, and strong mixing systems

In this section we prove Theorem 1.3. We start by establishing some properties of totally transitive points.

Definition 3.1. Given a totally transitive system (X, T), a point $x \in X$ is a *totally transitive point* if $\{T^{kn}x : n \in \mathbb{N}_0\}$ is dense in X for every $k \in \mathbb{N}$.

A transitive system always has a transitive point, but it is not immediately obvious that every totally transitive system has a totally transitive point. Nevertheless, the next lemma shows that this is indeed the case.

Lemma 3.2. *Let (X, T) be totally transitive. Then one of the following two mutually exclusive statements is true.*

1. *$T : X \to X$ is surjective and there is a residual set of totally transitive points.*
2. *There exists exactly one transitive point $x \in X$, which is an isolated point of X. In this case x is a totally transitive point and $(X \backslash \{x\}, T)$ is a totally transitive subsystem of (X, T).*

Proof. Suppose T is surjective, i.e. $TX = X$. If x is a transitive point of (X, T), then

$$\overline{\{T^n(Tx) : n \in \mathbb{N}_0\}} = T\overline{\{T^n x : n \in \mathbb{N}_0\}} = TX = X,$$

and so Tx is also a transitive point. In particular, whenever (X, T) is transitive and T is

10

surjective, there exists a dense set of transitive points. Surjectivity of T implies surjectivity of T^k, so the system (X, T^k) contains a dense set of transitive points.

If $x \in X$ is transitive for T^k, then for every non-empty open set $U \subset X$ there exists some $n \in \mathbb{N}_0$ with $x \in T^{-nk}U$, so the union $\bigcup_{n \in \mathbb{N}_0} T^{-nk}U$ contains every transitive point and hence is an open dense set. Taking now a countable basis $(U_\ell)_{\ell \in \mathbb{N}}$ for X, it follows that the intersection $\mathcal{T} := \bigcap_{\ell, k \in \mathbb{N}} \bigcup_{n \in \mathbb{N}_0} T^{-kn}U_\ell$ is a countable intersection of dense open sets and hence is residual. To finish the proof, notice that any point in $\mathcal{T}$ is totally transitive, since for every $k \in \mathbb{N}$ and every non-empty open set $U \subset X$ there exists $\ell \in \mathbb{N}$ with $U_\ell \subset U$ and hence there exists some $n \in \mathbb{N}_0$ such that $T^{nk}x \in U_\ell \subset U$.

(2) If (X, T) is a transitive system that is not surjective, then we claim that $U := X \backslash TX$ has exactly one element. Indeed, TX is compact, hence U is open. By transitivity, there exists a point $x \in X$ with a dense orbit, so $T^n x \in U$ for some $n \geqslant 0$; the definition of U forces $n = 0$ and hence $x \in U$. If there were some $y \in U$ with $y \neq x$ we could take $V \subset U$ to be a neighborhood of y not containing x. The orbit of x never enters V which is a contradiction, and this proves the claim that $U = \{x\}$.

Let $X_0 = X \backslash \{x\} = TX$. Let $k \in \mathbb{N}$ and $y = T^k x$. Then $y \in X_0$. Since x is a totally transitive point and x is isolated,

$$\overline{\{T^{kn}y : n \geqslant 0\}} = \overline{\{T^{kn}x : n \geqslant 1\}} = X_0,$$

showing that y is a transitive point for (X_0, T^k), and in particular that (X_0, T^k) is a transitive system. Since k is arbitrary, (X_0, T) is totally transitive.

Since $X_0 = TX$, for $k \geqslant 2$, we have $\{T^{kn}x : n \geqslant 1\} \subset TX_0$. Because the set in the left hand side is dense in X_0, TX_0 is dense in X_0. As both TX_0 and X_0 are closed, we conclude that $TX_0 = X_0$, i.e. T is surjective on X_0. $\qquad \square$

Lemma 3.3. *If (X, T) is totally transitive then every transitive point of (X, T) is a totally transitive point.*

Proof. If T is not surjective, then by Lemma 3.2, (X, T) has exactly one transitive point. Since (X, T) is totally transitive, this unique transitive point is also a totally transitive point.

Next, assume that T is surjective and let x be a transitive point of (X, T). Let $k \geqslant 2$ and take $Y = \overline{\{T^{kn}x : n \in \mathbb{N}_0\}}$. Since (X, T^k) is transitive, there exists a point $y \in X$ that is T^k-transitive. Since x is transitive, it follows that $Y \cup T^{-1}Y \cup \ldots \cup T^{-(k-1)}Y = X$ and hence $T^j y \in Y$ for some $j \in \{0, 1, \ldots, k-1\}$. Since y is T^k-transitive and T is surjective, $T^j y$ is also T^k-transitive. It follows that Y contains a T^k-transitive point of X, and is invariant under T^k, so $Y = X$, proving that x is T^k-transitive. $\qquad \square$

The next lemma is the main technical step towards proving Theorem 1.3.

Lemma 3.4. *Let $S \subset \mathbb{N}$ be a syndetic set and let $k \in \mathbb{N}$ be such that $S \cup (S-1) \cup \ldots \cup (S-(k-1)) \supset \mathbb{N}$. Let $h > k$ and $S_i := S \cap (h^2\mathbb{N} + [ih, (i+1)h))$ for $0 \leqslant i \leqslant h-1$. For any totally*

transitive system (X,T) *and transitive point* $x \in X$, *there exists* $0 \leqslant i < j \leqslant h - 1$ *such that*

$$\overline{\{T^n x : n \in S_i\}} \cap \overline{\{T^n x : n \in S_j\}} \neq \varnothing.$$

Proof. Since S_i is nonempty for all i, the conclusion is obviously true if (X,T) is the trivial one-point system. Now suppose (X,T) is a nontrivial, totally transitive system and let x be a transitive point of (X,T). By Lemma 3.3, the point x is a totally transitive point. We then have that $T^{a\mathbb{N}+b}x$ is dense in $X \backslash \{x\}$ for any $a, b \in \mathbb{N}$ (where, for any set $S \subset \mathbb{N}_0$, $T^S x := \{T^n x : n \in S\}$); this can be seen by analysing the two cases arising in Lemma 3.2 individually.

For $0 \leqslant i \leqslant h - 1$, let $X_i := \overline{T^{S_i} x}$. Note that for all $0 \leqslant i \leqslant h - 1$ and $0 \leqslant j \leqslant k - 1$,

$$h^2 \mathbb{N} + [ih, (i+1)h) - j \supset h^2 \mathbb{N} + ih.$$

Therefore, for $0 \leqslant i \leqslant h - 1$ and $0 \leqslant j \leqslant k - 1$,

$$S_i - j = (S \cap (h^2 \mathbb{N} + [ih, (i+1)h))) - j \supset (S - j) \cap (h^2 \mathbb{N} + ih).$$

And so,

$$S_i \cup (S_i - 1) \cup \ldots \cup (S_i - (k-1)) \supset (S \cup (S-1) \cup \ldots \cup (S-(k-1))) \cap (h^2 \mathbb{N} + ih)$$
$$= h^2 \mathbb{N} + ih.$$

It follows that

$$X_i \cup T^{-1} X_i \cup \ldots \cup T^{-(k-1)} X_i \supset \overline{T^{S_i} x} \cup \overline{T^{S_i - 1} x} \cup \ldots \cup \overline{T^{S_i - (k-1)} x}$$
$$\supset \overline{T^{h^2 \mathbb{N} + ih} x} \qquad (3.1)$$
$$\supset X \backslash \{x\}.$$

Let μ be a T-invariant measure on X. Since (X,T) is totally transitive and nontrivial, it is not a finite system, and since x is a transitive point, it is not a periodic point. Therefore, $\mu(\{x\}) = 0$ and $\mu(X \backslash \{x\}) = 1$. Then (3.1) implies $\mu(X_i) \geqslant 1/k$ for all $0 \leqslant i \leqslant h - 1$. Since $h > k$, there must be $0 \leqslant i < j \leqslant h - 1$ such that $X_i \cap X_j \neq \varnothing$ which is the desired conclusion. $\square$

We are ready to prove Theorem 1.3 whose statement is repeated here for convenience.

Theorem 1.3. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is a weak interpolation set of all orders for totally transitive systems.*
2. *S is an interpolation set for totally transitive systems.*
3. *S is an interpolation set for weak mixing systems.*
4. *S is an interpolation set for strong mixing systems.*
5. *S is not syndetic.*

Proof. We have the chain of implications among various classes of topological dynamical systems: Strong mixing $\Rightarrow$ Weak mixing $\Rightarrow$ Totally transitive. Therefore, the following directions are obvious: $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ and so it remains to prove $(1) \Rightarrow (5)$ and $(5) \Rightarrow (4)$.

$(1) \Rightarrow (5)$: For the sake of contradiction, suppose S is syndetic. Let $k \in \mathbb{N}$ be such that $\bigcup_{i=0}^{k-1}(S - i) = \mathbb{N}$. Fix an arbitrary integer $h > k$ and define $S_i := S \cap (h^2 \mathbb{N} + [ih, (i+1)h))$ for

$0 \leqslant i \leqslant h - 1$. Let $f : S \to \mathbb{R}$ be a bounded function that satisfies $f(n) = i$ for $n \in S_i$.

By Lemma 3.4, for any totally transitive system (X, T) and transitive point $x \in X$, there exists $0 \leqslant i < j \leqslant h - 1$ such that $\overline{T^{S_i} x} \cap \overline{T^{S_j} x} \neq \varnothing$. Hence, there does not exist $F \in C(X)$ such that $F(T^{S_i} x) = \{i\}$ and $F(T^{S_j} x) = \{j\}$. As a result, there is no $F \in C(X)$ such that the function $n \mapsto F(T^n x)$ extends f. In particular, S is not weak interpolation of order h, and so also not weak interpolation of all orders, for totally transitive systems.

(5) $\Rightarrow$ (4): Let $S \subset \mathbb{N}$ be a set which is not syndetic, take an arbitrary bounded function $f : S \to \mathbb{C}$ and let $K \subset \mathbb{C}$ be a compact set containing $f(S)$. We consider the full shift with (potentially infinite) alphabet K, defined as the topological dynamical system $(K^{\mathbb{N}_0}, \sigma)$ where $\sigma : (z_n)_{n \in \mathbb{N}_0} \mapsto (z_{n+1})_{n \in \mathbb{N}_0}$ is the left shift. Note that $K^{\mathbb{N}_0}$, equipped with the product topology is a compact metric space and that σ is continuous. The system $(K^{\mathbb{N}_0}, \sigma)$ is strong mixing and, in particular, transitive. Let $y \in K^{\mathbb{N}_0}$ be a transitive point. We think of points in $K^{\mathbb{N}_0}$ as functions $\mathbb{N}_0 \to K$.

Since S is not syndetic, then for every $n \in \mathbb{N}$ there is an interval $I_n := \{m_n + 1, m_n + 2, \ldots, m_n + n\}$ of length n which is disjoint from S. Extend $f : S \to K$ to a function $g : \mathbb{N}_0 \to K$ by letting $g|_{I_n} = y|_{\{1, \ldots, n\}}$. Then the orbit closure of g as an element of $K^{\mathbb{N}_0}$ contains y, and since y is a transitive point, so is g. Finally, let $F : K^{\mathbb{N}_0} \to \mathbb{C}$ be the projection onto the 0-th coordinate. If follows that $F(\sigma^n g) = g(n)$ for all $n \in \mathbb{N}_0$. In particular, $F(\sigma^n g) = f(n)$ for all $n \in S$, showing that S is an interpolation set for strongly mixing systems. $\qquad \square$

4. Minimal and totally minimal systems

In this section we prove Theorem 1.5 by combining the next two lemmas. The first lemma comes from [12].

Lemma 4.1 (Glasner-Weiss [12, Theorem 1 (Part 1)]). *Piecewise syndetic sets are not weak interpolation of order 2 for minimal systems.*

Remark 4.2. The result of Glasner and Weiss given in [12] asserts that any piecewise syndetic set is not an interpolation set for minimal systems. However, their proof yields the stronger conclusion after careful examination. Indeed, their proof involves constructing, for any piecewise syndetic set S, a function η with domain S for which $\eta(s)$ cannot be represented as $F(T^s x)$ for $s \in S$ and a minimal (X, T). This function η always has codomain $\{0, 1\}$, and so this immediately also demonstrates that S is not weak interpolation of order 2 for minimal systems.

Lemma 4.3. *Every non-piecewise syndetic set is an interpolation set for totally minimal systems.*

Proof. Let $S \subset \mathbb{N}$ be a non-piecewise syndetic set. Let $u : S \to \mathbb{C}$ be an arbitrary bounded function and let $K \subset \mathbb{C}$ be a compact set containing $u(S)$ and 0. We construct a compact set $X \subset K^{\mathbb{N}_0}$ iteratively via auxiliary sequences $m_k \in \mathbb{N}$, closed (thereby compact) sets $X_k \subset K^{m_k}$ and $X'_k \subset K^{m_k+1}$, and $w_k \in X_k$. Define $m_0 = 1$, $X_0 = K$, $X'_0 = K^2$, and $w_0 = 0$. Suppose now that $k \geqslant 0$ and that m_k, X_k, and w_k are defined. Since X_k, X'_k are compact, there exist 2^{-k}-dense

finite subsets T_k, T'_k of X_k and X'_k respectively (in the ℓ^∞ metric). Without loss of generality, we assume that $k!$ divides both $|T_k|$ and $|T'_k|$. Define m_{k+1} to be a multiple of $m_k(k+1)!$ with the property that every interval of length m_{k+1} contains a gap in S of length $4m_k^2(|T_k| + |T'_k|)$, i.e. for all I of length m_{k+1}, there exists a subinterval J of length $4m_k^2(|T_k| + |T'_k|)$ so that $J \cap S = \varnothing$. (We use here the fact that S is not piecewise syndetic, which is equivalent to the fact that for all n, gaps of length n occur syndetically in S, i.e. $\mathbb{N} \backslash S$ is thickly syndetic.) Note that by construction, $k!$ divides m_k for all k.

Define X_{k+1} to be the set of all strings w with length m_{k+1} which are concatenations of strings in X_k and X'_k and which have the following property: for each $0 \leqslant i < k!$, the set of concatenated elements of X_k whose first letter is at a location in w equal to i (mod $k!$) is 2^{-k}-dense in X_k, and the set of concatenated elements of X'_k whose first letter is at a location in w equal to i (mod $k!$) is 2^{-k}-dense in X'_k. Similarly define X'_{k+1}, with the only change being that elements of X'_{k+1} must have length $m_{k+1} + 1$.

We note that both X_{k+1} and X'_{k+1} are nonempty. Indeed, if $T_k = \{w_1^{(k)}, \ldots, w_{|T_k|}^{(k)}\}$, $T'_k = \{w_1'^{(k)}, \ldots, w_{|T'_k|}'^{(k)}\}$, and u, v are arbitrary elements of X_k, X'_k respectively, then the reader may check that the string

$$w_k w_k \ldots w_k (w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T'_k|}'^{(k)} v) \ldots (w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T'_k|}'^{(k)} v)$$

is in X_{k+1}, where $w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T'_k|}'^{(k)} v$ (which has length equal to 1 (mod $k!$)) is repeated m_k times and the number of w_k at the beginning is chosen to make the entire word have length m_{k+1}. (Note that the terminal portion has length $m_k (m_k|T_k| + (m_k + 1)(|T'_k| + 1)) < 4m_k^2(|T_k| + |T'_k|)$, which is less than m_{k+1}.) Similarly,

$$w_1'^{(k)} w_k w_k \ldots w_k (w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T'_k|}'^{(k)} v) \ldots (w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T'_k|}'^{(k)} v)$$

is in X'_{k+1}, where the first concatenated w_k above is replaced by $w_1'^{(k)}$.

Since w_k is a prefix of w_{k+1} for all k, we can define y to be the limit of the w_k, and define X to be the closure of the orbit of y.

We claim that for any $j > 0$, y is uniformly recurrent under the shift action by σ^j. Indeed, for any neighborhood U of y, there exists $k > j$ so that $V = \{z : \rho(z_1 \ldots z_{m_k}, y_1 \ldots y_{m_k}) < 2^{-k}\} \subset U$. (Here ρ denotes the metric in X_k.) By definition, $w_k = y_1 \ldots y_{m_k} \in X_k$. Also by definition, y is a concatenation of elements of X_{k+1} and X'_{k+1}, meaning that the set of locations t for which $y(t) \ldots y(t + m_{k+1} - 1) \in X_k$ or $y(t) \ldots y(t + m_{k+1}) \in X'_k$ is syndetic (with gaps bounded from above by $m_{k+1} + 1$). For each such t, there exists a location $s \equiv -t$ (mod $k!$) so that $\rho(y(t+s)y(t+s+1) \ldots y(t+s+m_k-1), w_k) < 2^{-k}$, i.e. $\sigma^{t+s} y \in V \subset U$. In addition, $t + s$ is a multiple of $k!$, and so a multiple of j. This implies that the set $\{n \in \mathbb{N} : \sigma^{jn} y \in U\}$ is syndetic. Since U was arbitrary, y is uniformly recurrent for σ^j. Since j was arbitrary, X is totally minimal.

It remains to construct $x_u \in X$ for which $x_u(s) = u(s)$ for all $s \in S$. The construction of x_u proceeds in steps, where it is continually assigned values in K on more and more of $\mathbb{N}$, and undefined portions are labeled by $*$. Formally, define $x^{(0)} \in (K \cup \{*\})^{\mathbb{N}_0}$ by $x^{(0)}(s) = u(s)$

14

for $s \in S$ and $*$ for all other locations. Note (for induction purposes) that $x^{(0)}$ is an infinite concatenation of elements of X_0 and blocks of $*$ of length $m_0 = 1$, and that $x^{(0)}$ contains $*$ exactly on the intervals $(im_0, (i+1)m_0]$ which are disjoint from S.

Now, suppose that $x^{(k)}$ has been defined as an infinite concatenation of elements of X_k, X_k', and blocks of $*$ of length m_k which contains $*$ on an interval $(im_k, (i+1)m_k]$ if and only if it is disjoint from S. We wish to extend $x^{(k)}$ to $x^{(k+1)}$ by changing some $*$ symbols to values in K. Consider any i for which $S \cap (im_{k+1}, \ldots, (i+1)m_{k+1}] \neq \varnothing$. The portion of $x^{(k)}$ occupying that interval is a concatenation of words in X_k and blocks of $*$ of length m_k. By definition of m_{k+1}, $(im_{k+1}, \ldots, (i+1)m_{k+1}]$ contains a subinterval J of length $4m_k^2(|T_k| + |T_k'|)$ which is disjoint from S. This must contain a subinterval of the form $(am_k, am_k + 1, \ldots, bm_k]$, where $b - a = m_k|T_k| + (m_k + 1)(|T_k'| + 1)$, which is filled with $*$. Replace this block of $*$ by

$$(w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T_k'|}'^{(k)} v) \ldots (w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T_k'|}'^{(k)} v),$$

where the string $w_1^{(k)} w_2^{(k)} \ldots w_{|T_k|}^{(k)} w_1'^{(k)} w_2'^{(k)} \ldots w_{|T_k'|}'^{(k)} v$ is repeated m_k times. Exactly as argued above for w_{k+1}, this string includes all elements of T_k and T_k' at locations starting at all residue classes modulo $k!$. Fill all other blocks of $*$ of length m_k in $x^{(k)}((im_{k+1}, \ldots, (i+1)m_{k+1}])$ with arbitrary elements of X_k.

By definition, this creates a word in X_{k+1}, which we denote by $w_i^{(k+1)}$. Define $x^{(k+1)}((im_{k+1}, (i+1)m_{k+1}]) = w_i^{(k+1)}$ for any i as above (i.e. those for which $S \cap (im_{k+1}, \ldots, (i+1)m_{k+1}] \neq \varnothing$) and as $*$ elsewhere. Note that $x^{(k+1)}$ is an infinite concatenation of words in X_{k+1}, X_{k+1}', and blocks of $*$ of length m_{k+1}, which contains $*$ on any interval $(im_{k+1}, (i+1)m_{k+1}]$ if and only if it is disjoint from S, and which agrees with $x^{(k)}$ on all locations where $x^{(k)}$ did not contain $*$.

Continue in this way to build a sequence $(x^{(k)})$. Since each is obtained from the previous by changing some $*$s to values in K (which are not changed in future steps), they approach a limit x_u. Since $S \neq \varnothing$, $S \cap (0, m_k] \neq \varnothing$ for all large enough k, and so $x^{(k)}((0, m_k])$ has no $*$, meaning that $x_u \in K^{\mathbb{N}_0}$. By definition, x_u agrees with $x^{(0)}$ on all locations where $x^{(0)}$ did not have $*$, and so $x_u(s) = u(s)$ for all $s \in S$. It remains only to show that $x_u \in X$. To see this, we note that for all sufficiently large k, $x_u(0, m_k] \in X_k$ by definition. Also by definition, for all sufficiently large k, y begins with w_{k+1}, which contains a subword within distance 2^{-k} of $x_u(0, m_k]$. This yields a sequence of shifts of y which converges to x_u, proving that $x_u \in X$ and $x_u(s) = u(s)$ for all $s \in S$.

Now, simply define $F \in C(X)$ by $F(x) = x(0)$. Then for all $s \in S$, $F(\sigma^s x_u) = x_u(s) = u(s)$. Since u was arbitrary and (X, σ) is totally minimal, we've shown that S is an interpolation set for totally minimal systems. $\qquad\square$

Theorem 1.5. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is a weak interpolation set of order 2 for minimal systems.*
2. *S is a weak interpolation set of all orders for minimal systems.*
3. *S is an interpolation set for minimal systems.*
4. *S is an interpolation set for totally minimal systems.*
5. *S is not piecewise syndetic.*

Proof. The implications $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ are obvious. The implication $(5) \Rightarrow (4)$ is proved in Lemma 4.3 and the implication $(1) \Rightarrow (5)$ follows from Lemma 4.1. $\qquad\square$

5. Uniquely ergodic and strictly ergodic systems

We prove Theorem 1.6 in this section. First, we observe that if $d^*(S) > 0$, then S is not a weak interpolation of all orders for uniquely ergodic systems.

Lemma 5.1. *If $d^*(S) > 0$, then S is not a weak interpolation set of all orders for uniquely ergodic systems.*

Proof. Suppose $d^*(S) > 1/k$ for some $k \in \mathbb{N}$. Let $(F_N)_{N \in \mathbb{N}}$ be a sequence of intervals in $\mathbb{N}$ such that

$$\lim_{N \to \infty} \frac{1}{|F_N|} \sum_{n \in F_N} 1_S(n) = d^*(S) > 1/k.$$

Without loss of generality, assume the F_N's are pairwise disjoint. Define the function $f : S \to \{0, 1, \ldots, k-1\}$ by $f(n) = i$ if $n \in F_N$ for some N satisfying $N \equiv i \bmod k$ and arbitrary if $n \notin \bigcup_{N \in \mathbb{N}} F_N$. We claim that there does not exist a uniquely ergodic system (X, T), a continuous function $F : X \to \mathbb{C}$, and a transitive point $x \in X$ such that $f(n) = F(T^n x)$ for all $n \in S$. As a result, S is not a weak interpolation set of all orders for the class of uniquely ergodic systems.

For the sake of contradiction, assume our claim is false and that such (X, T), x, and F exist. For each $0 \leqslant i < k$, define $X_i = F^{-1}(\{i\})$; then the sets X_i are compact, nonempty, and disjoint subsets of X. We may then, for $0 \leqslant i < k$, define nonnegative continuous functions $g_i : X \to [0, 1]$ for which $g_i = 1$ on X_i and $g_i = 0$ on X_j for $j \neq i$. Define $U_i = \{x \ : \ g_i(x) > 0\}$; then by definition the U_i are disjoint open sets.

For any fixed i and any $m \in \mathbb{N}$, by definition $\sigma^s x \in X_i$ for all $s \in S \cap F_{mk+i}$, and so $g_i(\sigma^s x) = 1$ for all such s. The sequence $(F_{mk+i})_{m \in \mathbb{N}}$ is still a sequence of intervals in $\mathbb{N}$ whose lengths tend to infinity, and so by unique ergodicity,

$$\lim_{m \to \infty} \frac{1}{|F_{mk+i}|} \sum_{n \in F_{mk+i}} g_i(\sigma^n x) = \int g_i \, d\mu.$$

We note that $g_i = 0$ on U_i^c, and so $\int g_i \, d\mu \leqslant \mu(U_i)$. Also, $g_i \geqslant 0$ and $g_i(\sigma^s x) = 1$ for $s \in S \cap F_{mk+i}$. This yields

$$\lim_{m \to \infty} \frac{|S \cap F_{mk+i}|}{|F_{mk+i}|} \leqslant \int g_i \, d\mu \leqslant \mu(U_i).$$

The limit on the left is $d^*(S)$, so we get $\mu(U_i) \geqslant d^*(S)$ for all i. However, there are k such sets, and $d^*(S) > 1/k$, so we have a contradiction. $\qquad\square$

The next proposition shows that if $d^*(S) = 0$, then S is an interpolation set for strictly ergodic systems.

Lemma 5.2. *Every set of zero Banach density is an interpolation set for strictly ergodic systems.*

Proof. Let $S \subset \mathbb{N}$ be a set satisfying $d^*(S) = 0$. Let $u : S \to \mathbb{C}$ be a bounded function and let $K \subset \mathbb{C}$ be a compact set containing $u(S)$ and 0. We construct $X \subset K^{\mathbb{N}_0}$ iteratively via auxiliary sequences $m_k \in \mathbb{N}$, compact sets $X_k \subset K^{m_k}$, and $w_k \in X_k$. Define $m_0 = 1$, $X_0 = K$, and $w_0 = 0$. Now suppose that m_k, X_k, and w_k are defined. Since X_k is compact, there exists a 2^{-k}-spanning subset T_k (in the ℓ^∞ metric). Since $d^*(S) = 0$, we can define m_{k+1} to be a multiple of $(2k+2)m_k$ larger than $(2k+2)m_k|T_k|$ with the property that for every interval $I \subset \mathbb{N}$ of length m_{k+1}, $\frac{|I \cap S|}{|I|} < ((2k+2)m_k)^{-1}$.

Define X_{k+1} to be the set of all concatenations of sets of $\frac{m_{k+1}}{m_k}$ strings in X_k with the following two properties: each $w \in T_k$ appears at least once, and w_k appears at least $(1 - (k+1)^{-1})\frac{m_{k+1}}{m_k}$ times. (This set is nonempty since $\frac{m_{k+1}}{m_k} \geqslant (k+1)|T_k|$.) Define w_{k+1} to be any element in X_{k+1} with w_k as a prefix. Since w_k is a prefix of w_{k+1} for all k, we can define y to be the limit of the w_k, and define X to be the closure of the orbit of y.

We claim that y is uniformly recurrent. Indeed, for any neighborhood U of y, there exists k so that $\{z \; : \; \rho(z_1 \ldots z_{m_k}, y_1 \ldots y_{m_k}) < 2^{-k}\} \subset U$. (Here ρ is the metric on X_k.) By definition, $w_k = y_1 \ldots y_{m_k} \in X_k$, and so there exists $v \in T_k$ within distance 2^{-k} of w_k. Finally, y is a concatenation of elements in X_{k+1}, each of which contains v by definition. So, the set of shifts of y in U is syndetic with gaps bounded by $2m_{k+1}$. Since U was arbitrary, y is uniformly recurrent, and so X is minimal.

Next, we claim that X is uniquely ergodic. First, we note that every $x \in X$ is a shift of an infinite concatenation of A_k-words for each k, and that each A_k-word is a concatenation of A_{k-1}-words in which at most a proportion of $1/k$ are not w_{k-1}. Therefore, if $z \in \mathbb{C}$ does not occur in w_{k-1}, the frequency of occurrences of z in x is at most $1/k$. It follows that if z does not occur in any w_k, then z has density 0 of occurrences in x. Since x was arbitrary, this means by the ergodic theorem that $\mu([z]) = 0$ for all invariant measures μ. (Here $[z]$ is the cylinder set $\{x \in X : x(0) = z\}$.)

Define W to be the countable set of all complex numbers appearing in some w_k, and consider any cylinder of the form $R = \prod_{i=1}^{m}([a_i, b_i] \times [c_i, d_i]) \times \prod_{m+1}^{\infty} K$, where all $a_i, b_i, c_i, d_i \notin W$. Choose any ergodic measure μ and any $x \in X$ generic[5] for μ. Note that since all endpoints are not in W, by earlier observation the boundary of R has μ-measure 0, and so

$$\mu(R) = \lim_{n \to \infty} \frac{1}{n} |\{0 \leqslant i < n \; : \; \sigma^i x \in R\}|.$$

Consider now $k \in \mathbb{N}$ such that $m_k > m$. We note that y is an infinite concatenation of elements in X_{k+1}, and since X_{k+1} is closed, the same is true for x; therefore $x((im_{k+1}, (i+1)m_{k+1}]) \in X_{k+1}$ for all k. We denote

$$N(w_k) = |\{0 < j \leqslant m_k - m \; : \; w_k(j) \ldots w_k(j + m - 1) \in \prod_{i=1}^{m}([a_i, b_i] \times [c_i, d_i])\}|.$$

---

[5] In a system (X, T), a point $x \in X$ is said to be *generic* for an ergodic measure μ if $\frac{1}{N}\sum_{n=1}^{N} F(T^n x) \to \int_X F \, d\mu$ for every continuous function $F : X \to \mathbb{C}$.

Then, since each word $x([im_{k+1}, (i+1)m_{k+1}))$ is a concatenation of elements of X_k containing at least $(1 - (k+1)^{-1})\frac{m_{k+1}}{m_k}$ occurrences of w_k, we see that

$$iN(w_k)(1 - (k+1)^{-1})\frac{m_{k+1}}{m_k} \leqslant |\{0 \leqslant j \leqslant im_{k+1} - m \ : \ \sigma^j x \in R\}|$$
$$\leqslant iN(w_k)(1 - (k+1)^{-1})\frac{m_{k+1}}{m_k} + i(k+1)^{-1}m_{k+1} + im_{k+1}\frac{m}{m_k}.$$

Here, the first term of the upper bound comes from visits to R within one of the w_k concatenated within some $x((jm_{k+1}, (j+1)m_{k+1}]) \in X_{k+1}$, and the remaining terms are just an upper bound on the number of possible visits to R at other locations. By dividing by $L = im_{k+1}$ and taking limits as $i \to \infty$ (and recalling that x is generic for μ), we get

$$(1 - (k+1)^{-1})\frac{N(w_k)}{m_k} \leqslant \mu(R) \leqslant (1 - (k+1)^{-1})\frac{N(w_k)}{m_k} + (k+1)^{-1} + \frac{m}{m_k}.$$

This estimate holds for all k, and so we can let $k \to \infty$ to see that $\mu(R) = \lim_k \frac{N(w_k)}{m_k}$ (and particular this limit exists). Since this quantity depends only on the sequence w_k, $\mu(R)$ does not depend on x. Since the collection of cylinders R generates the Borel σ-algebra, μ does not depend on x, and so X is uniquely ergodic.

It remains to construct $x_u \in X$ for which $x(s) = u(s)$ for all $s \in S$. The construction of x_u proceeds in steps, where it is continually assigned values in K on more and more of $\mathbb{N}$, and undefined portions are labeled by $*$. Formally, define $x^{(0)} \in (K \cup \{*\})^{\mathbb{N}_0}$ by $x^{(0)}(s) = u(s)$ for $s \in S$ and $*$ for all other locations. Note (for induction purposes) that $x^{(0)}$ is an infinite concatenation of elements of X_0 and blocks of $*$ of length $m_0 = 1$, and that $x^{(0)}$ consists of $*$ on any "interval" $(im_0, (i+1)m_0]$ which is disjoint from S.

Now, suppose that $x^{(k)}$ has been defined as an infinite concatenation of elements of X_k and blocks of $*$ of length m_k which consists of only $*$ on any interval $(im_k, (i+1)m_k]$ which is disjoint from S. We wish to extend $x^{(k)}$ to $x^{(k+1)}$ by changing some $*$ symbols to values in K. Consider any i for which $S \cap (im_{k+1}, \dots, (i+1)m_{k+1}] \neq \varnothing$. The portion of $x^{(k)}$ occupying that interval is a concatenation of elements of X_k and blocks of $*$ of length m_k. By definition of m_{k+1}, $|S \cap (im_{k+1}, \dots, (i+1)m_{k+1}]| < \frac{m_{k+1}}{(2k+2)m_k}$. This means that if $x^{(k)}((im_{k+1}, \dots, (i+1)m_{k+1}])$ is written as a concatenation of elements of X_k and blocks of $*$ of length m_k, at least $(1 - (2k+2)^{-1})\frac{m_{k+1}}{m_k}$ are blocks of $*$. Overwrite any $(1 - (k+1)^{-1})\frac{m_{k+1}}{m_k}$ of these with w_k, leaving at least $\frac{m_{k+1}}{(2k+2)m_k} \geqslant |T_k|$. Fill all remaining ones with elements of T_k in a way that each is used at least once.

By definition, this creates a word in X_{k+1}, which we denote by $w_i^{(k+1)}$. Define $x^{(k+1)}((im_{k+1}, (i+1)m_{k+1}]) = w_i^{(k+1)}$ for any i as above (i.e. those for which $S \cap (im_{k+1}, \dots, (i+1)m_{k+1}] \neq \varnothing$) and as $*$ elsewhere. Note that $x^{(k+1)}$ is an infinite concatenation of elements of X_{k+1} and blocks of $*$ of length m_{k+1} which contains $*$ on any interval $(im_{k+1}, (i+1)m_{k+1}]$ which is disjoint from S, and that $x^{(k+1)}$ agrees with $x^{(k)}$ on all locations where $x^{(k)}$ did not contain $*$.

Continue in this way to build a sequence $x^{(k)}$. Since each is obtained from the previous by changing some $*$s to values in K (which are not changed in future steps), they approach a limit x_u. Since $S \neq \varnothing$, $S \cap (0, m_k] \neq \varnothing$ for all large enough k, and so $x^{(k)}((0, m_k])$ has no $*$, meaning

that $x_u \in K^{\mathbb{N}_0}$. By definition, x_u agrees with $x^{(0)}$ on all locations where $x^{(0)}$ did not have $*$, and so $x_u(s) = u(s)$ for all $s \in S$. It remains only to show that $x_u \in X$. To see this, we note that for all sufficiently large k, $x_u(0, m_k] \in X_k$ by definition. Also by definition, for all sufficiently large k, y begins with w_{k+1}, which contains a subword in T_k which is within distance 2^{-k} of $x_u(0, m_k]$. This yields a sequence of shifts of y which converges to x_u, proving that $x_u \in X$ and $x_u(s) = u(s)$ for all $s \in S$. Finally, this implies that S is interpolation for strictly ergodic systems by considering $F \in C(X)$ defined by $F(x) = x(0)$. $\qquad\square$

We are ready to prove Theorem 1.6 whose statement is repeated here for convenience.

Theorem 1.6. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is a weak interpolation set of all orders for uniquely ergodic systems.*
2. *S is an interpolation set for uniquely ergodic systems.*
3. *S is an interpolation set for strictly ergodic systems.*
4. *$d^*(S) = 0$.*

Proof. It is obvious that $(3) \Rightarrow (2) \Rightarrow (1)$. The implication $(4) \Rightarrow (3)$ follows from Lemma 5.2 and $(1) \Rightarrow (4)$ follows from Lemma 5.1. $\qquad\square$

Lastly, we prove Proposition 1.7; we recall the statement here for convenience.

Proposition 1.7. *If $S \subset \mathbb{N}$ is syndetic and the word complexity of the sequence 1_S grows subexponentially (i.e. the orbit closure of 1_S under the left shift has zero entropy), then S is not a weak interpolation set of order 2 for uniquely ergodic systems.*

Proof. Suppose for a contradiction that a set S exists which is a weak interpolation set of order 2 for uniquely ergodic systems and which is syndetic and has subexponential word complexity. Say that S is syndetic with gaps less than G, and then for every interval I of length at least $2G$, $|S \cap I| > (2G)^{-1}|I|$.

For any $n > 2G$, denote by $p(n)$ the word complexity of χ_S, i.e. the number of different sets which occur as $(S \cap [k, k + n)) - k$ for $k \in \mathbb{N}$. By subadditivity of upper density, there exists such a set $J = \{j_1, \dots, j_{|J|}\}$ which occurs with upper density at least $(p(n))^{-1}$, i.e. $A = \{k : (S \cap [k, k + n)) - k = J\}$ has upper density at least $(p(n))^{-1}$. By partitioning A by residue classes modulo n, we can pass to a subset B of A with gaps of at least n and upper density at least $(np(n))^{-1}$. It is easy to partition B into subsets B_i, $1 \leqslant i \leqslant 2^{|J|}$, with the same upper density. Note that $|J| > (4G)^{-1}n$ by definition of N.

Enumerate the words in $\{0, 1\}^{|J|}$ as $w_1, \dots, w_{2^{|J|}}$. Define $y : S \to \{0, 1\}$ by assigning, for every $1 \leqslant i \leqslant 2^{|J|}$, every $m \in B_i$, and every $1 \leqslant k \leqslant |J|$, $y(m + j_k) = w_i(k)$, and defining y arbitrarily otherwise. This is well-defined since we assumed B to have gaps of at least $n \geqslant |J|$. By assumption, there exists a uniquely ergodic system (X, T), transitive point x, and $f \in C(X)$ so that $f(T^x x) = y(s)$ for all $s \in S$.

Denote the unique measure of X by μ. For any $1 \leqslant i \leqslant 2^{|J|}$, any $m \in B_i$, and any $1 \leqslant k \leqslant |J|$, $f(\sigma^{m+j_k} x) = y(m + j_k) = w_i(k)$. Therefore, for each such m, $\sigma^m(x) \in \bigcap_{k=1}^{|J|} \sigma^{-j_k} f^{-1}\{w_i(k)\}$;

19

denote the latter set by C_i. By unique ergodicity and the fact that B_i has upper density at least $(np(n))^{-1}$, $\mu(C_i) \geqslant (np(n))^{-1}$. However, the sets C_i are disjoint, so $2^{|J|}(np(n))^{-1} \leqslant 1$. Since $|J| > (4G)^{-1}n$, we contradict subexponential growth of $p(n)$ for large enough n. Our original hypothesis was false, and S is not a weak interpolation set of order 2 for uniquely ergodic systems. $\qquad\square$

6. Systems of bounded entropy

Proposition 1.8. *Let $S \subset \mathbb{N}$. The following are equivalent:*

1. *S is an interpolation set for systems of finite entropy.*
2. *S is an interpolation set for systems of entropy $\leqslant M$ for some fixed $M \geqslant 0$.*
3. *S is an interpolation set for systems of zero entropy.*
4. *$d^*(S) = 0$.*

Proof. It is obvious that $(3) \Rightarrow (2) \Rightarrow (1)$ and so it remains to prove $(4) \Rightarrow (3)$ and $(1) \Rightarrow (4)$.

$(4) \Rightarrow (3)$: Consider any set S with $d^*(S) = 0$ and any bounded $f : S \to \mathbb{C}$. Extend f to $\tilde{f} : \mathbb{N}_0 \to \mathbb{C}$ by defining $\tilde{f} = 0$ on $\mathbb{N}_0 \backslash S$. Now, the orbit closure of $\tilde{f}$ under the left shift σ is a shift system (X, σ) with $X \subset \mathbb{C}^{\mathbb{N}_0}$. We claim that (X, σ) has zero entropy. Defining $F(x) = x(0)$ for all x we have that for $n \in S$, $F(\sigma^n \tilde{f}) = \tilde{f}(n) = f(n)$, and $\tilde{f}$ is a transitive point in X by definition, so assuming the claim we have that S is an interpolation set for systems of zero entropy.

We claim that every $x \in X$ satisfies $d^*(x^{-1}(\mathbb{C}\backslash\{0\})) = 0$. Indeed, for any $\varepsilon > 0$, by definition of upper Banach density, there exists N so that for every $i \in \mathbb{N}$,

$$\frac{|\{0 \leqslant j < N \; : \; \tilde{f}(i+j) \neq 0\}|}{N} < \varepsilon.$$

This property is preserved under shifts and limits, and so holds for any $x \in X$ as well, implying that $d^*(x^{-1}(\mathbb{C}\backslash\{0\})) = 0$. Therefore, by the ergodic theorem, the only invariant measure on (X, σ) is the delta-measure for the constant sequence 0. By the variational principle, (X, σ) has topological entropy 0.

$(1) \Rightarrow (4)$: Consider any set S with $d^*(S) > 0$ and assume for a contradiction that S is an interpolation set for the class of finite entropy systems. Choose $\delta < d^*(S)$.

There exist arbitrarily large $m \in \mathbb{N}$ and sets $S_m \subset [1, m]$ with $|S_m| \geqslant \delta m$ for which infinitely many shifts of S_m are contained in S. Write $S_m = \{s_{m,1}, \ldots, s_{m,|S_m|}\}$ in increasing order for all m. Let $\mathcal{S}^1$ denote the unit circle $\{z \in \mathbb{C} : |z| = 1\}$. Construct a sequence $f : S \to \mathcal{S}^1$ which, for all k and m, contains every possible sequence of kth roots of unity of length $|S_m|$ along some shift of S_m, i.e. for any such string $c = (c_1, \ldots, c_{|S_m|})$, there exists j_c so that $f(j_c + s_i) = c_i$ for $1 \leqslant i \leqslant |S_m|$. By assumption, there exists a finite entropy system (X, T), $x \in X$, and $F \in C(X)$ where $F(T^n x) = f(n)$ for all n. Say that $h(X, T) < \delta \log k$. Now, define ε so that points of X within distance ε cannot have F-values which are distinct kth roots of unity. For every m and every $|S_m|$-tuple $c = (c_1, \ldots, c_m)$ of kth roots of unity, $F(\sigma^{j_c + s_i} x) = f(j_c + s_i) = c_i$ for $1 \leqslant i \leqslant |S_m|$. This implies that the collection $\{\sigma^{j_c} x\}$ of cardinality $k^{|S_m|} \geqslant k^{\delta m}$ is (m, ε)-

separated, and since m can be arbitrarily large, $h(X,T) \geqslant \delta \log k$, a contradiction. $\qquad\square$

Proposition 1.9. *Every subset $S \subset \mathbb{N}$ is a weak interpolation set of all orders for systems of finite entropy.*

Proof. We will prove that $\mathbb{N}$ itself is a weak interpolation set of all orders for the class of systems of finite topological entropy. Then, clearly, the same will be true for all subsets of $\mathbb{N}$.

Consider any k and any function $f : \mathbb{N}_0 \to \{0, \ldots, k-1\}$. The orbit closure of f under the left shift is a subshift (X, σ) that has entropy bounded by $\log k$. Define $F : X \to \{0, \ldots, k-1\}$ by $F(x) = x(0)$ for all $x \in X$. It follows that f is a transitive point of (X, σ) and $F(\sigma^n f) = f(n)$ for all $n \in \mathbb{N}$. $\qquad\square$

Recall the entropy function $H : [0,1] \to \mathbb{R}$ with $H(\delta) = -\big(\delta \log \delta + (1-\delta) \log(1-\delta)\big)$ for $\delta \in (0,1)$ and $H(0) = H(1) = 0$. The function H is known to be the exponential growth rate of the number of words with proportion of fewer than δ non-0 symbols on a 2-letter alphabet; we include the proof of the following corresponding estimate on larger alphabets for completeness.

Lemma 6.1. *Fix $k \geqslant 2$. For each $m \in \mathbb{N}$ and $\delta \in [0, 1/2]$, let*

$$S(m, \delta, k) := \Big\{ f : [m] \to \{0, \ldots, k-1\} : \big|f^{-1}(0)\big| \geqslant (1-\delta)m \Big\}.$$

Then

$$\lim_{m \to \infty} \frac{\log \big|S(m, \delta, k)\big|}{m} = \inf_m \frac{\log \big|S(m, \delta, k)\big|}{m} = H(\delta) + \delta \log(k-1).$$

Proof. Every element of $S(m, \delta, k)$ can be described by first describing the locations of the non-0 letters, and then assigning any non-0 letter at each of those locations. Therefore,

$$|S(m, \delta, 2)| = \sum_{i=0}^{\lfloor \delta m \rfloor} (k-1)^i \binom{m}{i}.$$

Since $\delta \leqslant 1/2$, $\binom{m}{i}$ is increasing from $i = 0$ to $\lfloor \delta m \rfloor$ (and $(k-1)^i$ is increasing for all i), and so

$$(k-1)^{\lfloor \delta m \rfloor} \binom{m}{\lfloor \delta m \rfloor} \leqslant |S(m, \delta, 2)| \leqslant (\lfloor \delta m \rfloor + 1)(k-1)^{\lfloor \delta m \rfloor} \binom{m}{\lfloor \delta m \rfloor}. \qquad (6.1)$$

Define $\delta_m = \frac{\lfloor \delta m \rfloor}{m}$. Clearly $\delta_m \leqslant 1/2$ and $\delta_m \to \delta$. By Stirling's approximation,

$$
\begin{aligned}
(k-1)^{\lfloor \delta m \rfloor} \log \binom{m}{\lfloor \delta m \rfloor} &= (k-1)^{\delta_m m} \log \binom{m}{\delta_m m} \\
&= \delta_m m \log(k-1) + \log m! - \log(\delta_m m)! - \log((1-\delta_m)m)! \\
&= \delta_m m \log(k-1) + m \log m - m - (\delta_m m) \log(\delta_m m) \\
&\quad + \delta_m m - ((1-\delta_m)m) \log((1-\delta_m)m) + (1-\delta_m)m + o(m) \\
&= \delta_m m \log(k-1) + m H(\delta_m) + o(1).
\end{aligned}
$$

Since H is a continuous function and $\delta_m \to \delta$, by (6.1), $\frac{\log |S(m, \delta, k)|}{m} \to H(\delta) + \delta \log(k-1)$.

Finally, it's easily checked that the sequence $\log |S(m, \delta, k)|$ is subadditive in k, and so by Fekete's subadditivity lemma, the limit of $\frac{\log |S(m, \delta, k)|}{m}$ is also the infimum. $\qquad\square$

We recall Theorem 1.10 from the introduction.

Theorem 1.10. *Let $\delta \in [0, 1/2]$. For any $k \in \mathbb{N}$ and a set $S \subset \mathbb{N}$, we have the following:*

1. *If $d^*(S) > \delta$, then S is not a weak interpolation set of order k for systems of entropy $\leqslant \delta \log k$.*

2. *If $d^*(S) \leqslant \delta$, then S is a weak interpolation set of order k for systems of entropy $\leqslant H(\delta) + \delta \log(k - 1)$.*

Moreover, (1) is sharp in the following sense:

3. *There exists $S \subset \mathbb{N}$ with $d^*(E) = \delta$ such that S is a weak interpolation set of order k for the class of systems of entropy $\leqslant \delta \log k$.*

Proof. (1) Assume $d^*(S) > \delta$ and assume that S is a weak interpolation set of order k for some system (X, T). Then there exist arbitrarily large $m \in \mathbb{N}$ and infinitely many intervals $[n + 1, n + m]$ such that

$$|S \cap [n + 1, n + m]| \geqslant d^*(S)m.$$

Since there are only finitely many subsets of $[1, m]$, there exists some set $S_m \subset [1, m]$ for which infinitely many shifts are contained in S. Construct a sequence $f : S \to \{0, 1, \ldots, k - 1\}$ which, for all m, contains all possible functions on $S_m \to \{0, 1, \ldots, k-1\}$ up to shift; note that there are at least $k^{d^*(S)m}$ many such words. By assumption, there exists transitive $x_f \in X$ and $\phi \in C(X)$ with $\phi(T^n x_f) = f(n)$ for all $n \in S$. Note that the sets $S_i = \phi^{-1}(\{i\})$ are disjoint compact sets for $0 \leqslant i < k$, and so they are pairwise separated by some distance ε. Then, for all m, by construction, there is an (m, ε)-separated set of cardinality $k^{d^*(S)m}$, consisting of iterates $T^n x_f$ for $n \in S$. Since x_f is transitive, this implies that $h(X) \geqslant d^*(S) > \delta$, a contradiction.

(2) Suppose $S \subset \mathbb{N}$ such that $d^*(S) \leqslant \delta \leqslant 1/2$ and that $f : S \to \{0, \ldots, k - 1\}$. Define $x_f : \mathbb{N}_0 \to \{0, \ldots, k - 1\}$ by sending all elements of S^c to 0, and then define a subshift X_f as the orbit closure of x_f under the left shift map.

For any $\delta' > \delta$, by definition of d^* there exists M so that $|S \cap I| < |I|\delta'$ for all intervals of length at least M. Therefore, for all $m \geqslant M$, $L_m(X_f) \subset S(m, \delta', k)$, meaning that by Lemma 6.1, $h(X_f) \leqslant H(\delta') + \delta' \log(k - 1)$. Since $\delta' > \delta$ was arbitrary and H is continuous, $h(X_f) \leqslant H(\delta) + \delta \log(k - 1)$, completing the proof.

(3) The idea is to choose S very regular so that even though S has large density, we can always extend a function on S to a function on $\mathbb{N}$ that has small entropy.

Let $S = \{\lfloor n/\delta \rfloor : n \in \mathbb{N}\}$. Then $d^*(S) = \delta$ by definition. Define $x = \chi_S$; then the orbit closure X of x under the left shift map is a so-called Sturmian subshift with rotation number δ. It's known that the number of m-letter words in the language of any such subshift is $m + 1$ for all m, and it's clear by definition that the number of 1s in any word of length m in the language of X is at most $\lceil m\delta \rceil$ where $\lceil \cdot \rceil$ denotes the ceiling function.

Now, further define a subshift Y as the set of all 0-1 sequences obtained by changing some 1s in a point of X to any letters in $\{0, \ldots, k - 1\}$. The number of words of any length m in Y is at most $(m + 1)k^{\lceil m\delta \rceil}$, so $h(Y) \leqslant \delta \log k$.

Finally, for any function $f : S \to \{0, 1, \ldots, k-1\}$, we can extend f to $x : \mathbb{N}_0 \to \{0, 1, \ldots, k-1\}$ by mapping to 0 on S^c. The orbit closure of f under the left shift is a subshift X_f for which x is a transitive point, and X_f is contained in Y, so $h(X_f) \leqslant h(Y) \leqslant \delta \log k$. $\qquad \square$

7. Interpolation sets for distal systems and connections to pointwise recurrence

A topological system (X, T) on a metric space X with metric ρ is called *distal* if for all distinct $x, y \in X$ one has $\inf_{n \in \mathbb{N}} \rho(T^n x, T^n y) > 0$. The class of distal systems is closed under taking products and inverse limits. It is a subclass of the class of zero entropy systems and it contains all rotations on compact abelian groups and, more generally, all nilsystems. Distal systems have been studied in ergodic theory for a long time, partly because their measure theoretic counterparts are core components in the structure theory of measure preserving systems (see Furstenberg [8] and Zimmer [38, 39]).

Regarding interpolation sets for distal systems, it is shown in [29, Corollary 5.1] that any set of positive integers $S = \{s_1 < s_2 < \ldots\}$ satisfying

$$\limsup_{n \in \mathbb{N}} \frac{\log(s_{n+1})}{\log(s_{n+1} - s_n)} < \infty \tag{7.1}$$

is an interpolation set for distal systems. (In fact, in this case, S is shown to be an interpolation set for a special subclass of distal systems called skew products on tori.) The following natural question asked in [25] is still open.

Question 7.1. *Let $S \subset \mathbb{N}$ be an interpolation set for distal systems and $F \subset \mathbb{N}$ finite. Is it true that $S \cup F$ is an interpolation set for distal systems?*

Note that Question 7.1 has an affirmative answer if the class of distal systems is replaced with rotations on compact abelian groups (cf. [24]), nilsystems (cf. [25]), totally transitive systems (cf. Theorem 1.3), weak mixing systems (cf. Theorem 1.3), strong mixing systems (cf. Theorem 1.3), minimal systems (cf. Theorem 1.5), or uniquely ergodic systems (cf. Theorem 1.6). This prompts us to believe that the answer for distal systems is also positive.

There exists a noteworthy connection between Question 7.1 and the notion of pointwise recurrence in distal systems. A set $R \subset \mathbb{N}$ is a *set of pointwise recurrence for distal systems* if for every distal system (X, T) and every point $x \in X$, we have $\inf_{n \in R} \rho(x, T^n x) = 0$.

Proposition 7.2. *If every set of pointwise recurrence for distal systems can be partitioned into two disjoint sets of pointwise recurrence for distal systems, then the answer to Question 7.1 is positive.*

Proof. Let S be an interpolation set for distal systems and let $a \in \mathbb{N}\backslash S$. First, we show that $S - a$ is not a set of pointwise recurrence for distal systems. Indeed, for contradiction, suppose the opposite. Then by our hypothesis, $S - a$ can be partitioned into two sets A and B both of which are sets of pointwise recurrence for distal systems. Therefore, for any distal system

23

(X, T), any point $x \in X$,

$$x \in \overline{\{T^n x : n \in A\}} \cap \overline{\{T^n x : n \in B\}}.$$

(Here we use the fact that distal systems are semisimple, i.e. the orbit closure of every point is minimal.) It follows that

$$T^a x \in \overline{\{T^n x : n \in A + a\}} \cap \overline{\{T^n x : n \in B + a\}}. \tag{7.2}$$

Note that $A + a$ and $B + a$ are disjoint subsets of S and so we can define a function $f : S \to \{0, 1\}$ with $f|_{A+a} = 0$ and $f|_{B+a} = 1$. Because the intersection in (7.2) is nonempty for any distal system (X, T) and any point $x \in X$, the function f cannot be extended to a sequence coming from a distal system. This contradicts our assumption that S is an interpolation set for distal systems.

To show $S \cup \{a\}$ is an interpolation set for distal systems, let $g : S \cup \{a\} \to [0, 1]$ be an arbitrary function. Since S is an interpolation set for distal systems, there exists a distal system (X, T), a point $x_0 \in X$ and a function $F \in C(X)$ such that $F(T^n x_0) = g(n)$ for all $n \in S$. Because $S - a$ is not a set of pointwise recurrence for distal systems, there exists a distal system (Y, S) and a point $y_0 \in Y$ such that $y_0 \notin \overline{\{S^n y_0 : n \in S - a\}}$, and so $S^a y_0 \notin \overline{\{S^n y_0 : n \in S\}}$. Considering the product system $(X \times Y, T \times S)$ and the point (x_0, y_0), we have

$$(T \times S)^a (x_0, y_0) \notin \overline{\{(T \times S)^n (x_0, y_0) : n \in S\}}.$$

Therefore, there exists a continuous function $G : X \times Y \to [0, 1]$ such that

$$G(x, y) = \begin{cases} F(x), \text{ if } (x, y) \in \overline{\{(T \times S)^n (x_0, y_0) : n \in S\}}, \\ g(a), \text{ if } (x, y) = (T \times S)^a (x_0, y_0). \end{cases}$$

It follows that

$$G((T \times S)^n (x_0, y_0)) = \begin{cases} F(T^n x_0) = g(n), \text{ if } n \in S, \\ g(a), \text{ if } n = a. \end{cases}$$

Since $g : S \cup \{a\} \to [0, 1]$ is arbitrary, we conclude that $S \cup \{a\}$ is an interpolation set for distal systems. $\qquad \square$

In light of Proposition 7.2, the following open question arises naturally and warrants further investigation.

Question 7.3. *Is it true that every set of pointwise recurrence for distal systems can be partitioned into two sets of pointwise recurrence for distal systems?*

We remark that the analogues of Question 7.3 for compact abelian group rotations and nilsystems were answered in [24, 25, 31].

A set $E \subset \mathbb{N}$ is an *IP-set* if it contains an infinite sequence and all its finite sums, that is, there are $n_1 < n_2 < \ldots \in \mathbb{N}$ with $\{n_{i_1} + \ldots + n_{i_k} : k \in \mathbb{N}, i_1 < \ldots < i_k\} \subset E$. It is well known that IP-sets are sets of pointwise recurrence in distal systems [9, Theorem 9.11]. The following

question was asked by Host, Kra, and Maass in [18] and an affirmative answer to it would imply an affirmative answer to Question 7.3, because every IP-set can be partitioned into two disjoint IP-sets.

Question 7.4 (Host-Kra-Maass [18, Question 3.11])**.** *Is it true that every set of pointwise recurrence for distal systems is an IP-set?*

While we don't know how to answer Question 7.3, our last result shows that the stronger Question 7.4 has a negative answer.

Theorem 7.5. *There exists a set $F \subset \mathbb{N}$ that is a set of pointwise recurrence for distal systems but for any $x, y \in F$ we have $x + y \notin F$. In particular, F is not an IP-set.*

Proof. First we show that if $F \subset \mathbb{N}$ and $F - n$ contains an IP-set for all $n \in \mathbb{N}$, then F is a set of pointwise recurrence for distal systems. Indeed, let (X, T) be a distal system, $U \subset X$ be open and $x \in U$. Since the orbit closure of every point in a distal system is minimal, there is $n \in \mathbb{N}$ such that $T^n x \in U$. Since $F - n$ contains an IP-set, by Furstenberg's theorem [9, Theorem 9.11], there exists $m \in F - n$ such that $T^m(T^n x) \in U$. It follows that $T^{m+n} x \in U$ and note that $m + n \in F$.

Now we will construct the set F. Let $I_1, I_2, I_3, \ldots$ be pairwise disjoint infinite subsets of $\mathbb{N}$ such that $\min\{k : k \in I_n\} \geqslant n$ for all n. Let J_n be the IP-set generated by $\{10^k : k \in I_n\}$. Then the sets $J_1, J_2, J_3, \ldots$ are pairwise disjoint and every element of J_n is divisible by 10^n.

Let $F = \bigcup_{n \in \mathbb{N}}(J_n + n)$. Then for every $n \in \mathbb{N}$, the shift $F - n$ contains the IP-set J_n and hence F is a set of pointwise recurrence for distal systems. It remains to show that F does not contain a triple of the form $x, y, x + y$. By contradiction, assume there are n, m, k (not necessarily distinct) such that

$$\big((J_n + n) + (J_m + m)\big) \cap (J_k + k) \neq \varnothing. \tag{7.3}$$

In other words, there are $1 \leqslant n_1 < n_2 < \ldots < n_s$, $1 \leqslant m_1 < m_2 < \ldots < m_t$ and $1 \leqslant k_1 < k_2 < \ldots < k_u$ such that

$$(10^{n_1} + \ldots + 10^{n_s} + n) + (10^{m_1} + \ldots + 10^{m_t} + m) = 10^{k_1} + \ldots + 10^{k_u} + k. \tag{7.4}$$

By removing the terms that appear in both sides, we can assume that $\{k_1, \ldots, k_u\}$ is disjoint from $\{n_1, \ldots, n_s, m_1, \ldots, m_t\}$. In particular, k_u must be strictly larger than both n_s and m_t, for if, say, n_s was the larger, then because $k \leqslant k_1 < 10^{k_1}$, we would have

$$10^{n_s} > 10^{k_1} + \ldots + 2 \cdot 10^{k_u} > 10^{k_1} + \ldots + 10^{k_u} + k,$$

which contradicts (7.4). That leaves the only possibility that k_u is the largest. But this is also impossible since

$$10^{k_u} > 2(1 + 10^1 + \ldots + 10^{k_u - 1}) > (10^{n_1} + \ldots + 10^{n_s} + n) + (10^{m_1} + \ldots + 10^{m_t} + m).$$

Thus, the intersection in (7.3) is empty and hence F does not contain a triple of the form $x, y, x + y$. $\qquad \square$

8. Open questions

In this section, we list some natural questions that arise from our study and may be of interest to readers.

The first question is concerned with interpolation sets for compact abelian group rotations. A *compact abelian group rotation* (or *group rotation* for short) has the form (G, g) where G is a compact abelian group and $g \in G : x \mapsto g \cdot x$ for $x \in G$. It was shown in [35] that lacunary sets are interpolation sets for group rotations. On the other hand, no subexponential sets are interpolation sets for this class of systems [24]. For the sets S that are neither lacunary nor subexponential, whether S is an interpolation set for group rotations hinges on some delicate arithmetic properties instead of mere density. For example, $\{2^n : n \in \mathbb{N}\} \cup \{2^n + 2n : n \in \mathbb{N}\}$ is not interpolation while $\{2^n : n \in \mathbb{N}\} \cup \{2^n + 2n + 1 : n \in \mathbb{N}\}$ is. Based on these facts, it is desirable to have a complete combinatorial characterization of interpolation sets for group rotations:

Problem 8.1. *Give a combinatorial characterization of interpolation sets for compact abelian group rotations.*

Group rotations is a subclass of inverse limits of nilsystems, and so our next two questions consider interpolation sets for the latter, more general family of systems. For $k \in \mathbb{N}$, *k-step nilsystem* is a topological dynamical system (X, T) where $X = G/\Gamma$ for G a k-step nilpotent Lie group and Γ a discrete co-compact subgroup of G, and $T : X \to X$ is left translations by a fixed group element $g \in G$. Because a group rotation is an inverse limit of 1-step nilsytems, an interpolation set for group rotations is interpolation for inverse limits of k-step nilsystems for any $k \geqslant 1$. However, it was shown in [24] that there is an interpolation set for inverse limits of 2-step nilsystems which is not interpolation for group rotations. It is unknown if the same feature continues to hold for higher step nilsystems:

Question 8.2. *For $k \geqslant 2$, does there exist an interpolation set for inverse limits of $(k+1)$-step nilsystems that is not an interpolation set for inverse limits of k-step nilsystems?*

Remark that Question 8.2 was also asked in [24, 25].

Similarly to the situation with group rotations, we have a relatively good understanding of interpolation sets for inverse limits of nilsystems in terms of density [2, 25, 35]: if S is lacunary, S is an interpolation for inverse limits of nilsystems; if S is subexponential, S is not interpolation for this class. However, this characterization does not cover all subsets of $\mathbb{N}$ and so a complete characterization of interpolation sets of inverse limits of nilsystems is still missing:

Problem 8.3. *For $k \geqslant 2$, give a characterization for interpolation sets of inverse limits of k-step nilsystems.*

Next, we shift our focus to interpolation sets for distal systems. The orbit closure of every point in a distal system is minimal and therefore, by Theorem 1.5, an interpolation set for distal systems must be non-piecewise syndetic.

Moreover it follows from (7.1) that any sequence s_n for which the difference between consecutive elements $s_{n+1} - s_n$ grows sufficiently fast (e.g. $\{\lfloor n^{1+\varepsilon} \rfloor : n \in \mathbb{N}\}$ for any $\varepsilon > 0$), yields an interpolation set for distal systems. This observation naturally leads to the following question, aiming to weaken the condition (7.1):

Question 8.4. *Is it true that any set $S = \{n_1 < n_2 < \ldots\}$ for which the gaps $n_{k+1} - n_k$ tend to infinity is an interpolation set for distal systems?*

An important aspect, both in the condition (7.1) and in Question 8.4, is the emphasis on the growth of gaps $n_{k+1} - n_k$ rather than the growth of the sequence n_k itself. Controlling the growth rate of n_k is not sufficient for interpolation in distal systems. Indeed, one can construct examples of sets of arbitrarily fast growth rate that fail to be interpolation sets for distal systems by considering sparse IP-sets. Yet IP-sets do not provide a counterexample to Question 8.4 since they possess gaps that appear infinitely often.

We also have the following, more concrete, question about distal systems involving prime numbers.

Question 8.5. *Is it true that the set of primes $\mathbb{P}$ is an interpolation set for distal systems?*

According to the celebrated Zhang's theorem [37], the sequence of gaps between consecutive primes does not go to infinity. Therefore, Question 8.5 is not a special case of Question 8.4. In order for $\mathbb{P}$ to be an interpolation set for distal systems, for any partition $\mathbb{P} = A \cup B$, there must be a distal system (X, T) and a point $x \in X$ such that the closures $\overline{\{T^p x : p \in A\}}$ and $\overline{\{T^p x : p \in B\}}$ are disjoint. Related to this, recently, Kanigowski, Lemańczyk, and Radziwiłł [20] proved that the prime number theorem holds for skew products on $\mathbb{T}^2$ with $T(x, y) = (x + \alpha, y + h(x))$ where h is an analytic function. More precisely, they showed that for these systems (X, T) and for any $x \in X$, the set $\{T^p x : p \in \mathbb{P}\}$ is uniformly distributed on $\mathbb{T}^2$. That means for any partition $\mathbb{P} = A \cup B$,

$$\overline{\{T^p x : p \in A\}} \cup \overline{\{T^p : p \in B\}} = \mathbb{T}^2$$

and so these two closures cannot be disjoint. As a result, $\mathbb{P}$ is not interpolation for the skew products on $\mathbb{T}^2$ with analytic skewing functions. That being said, since the class of distal systems is much larger than the aforementioned skew products, the answer to Question 8.5 may be positive.

The next question is concerned with the threshold for density of weak interpolation sets for uniquely ergodic systems. It follows from the proof of [36, Theorem 8.1] that for any $\varepsilon > 0$, there exists a set S which is a weak interpolation set of order 2 for uniquely ergodic systems such that $d^*(S) > 1/2 - \varepsilon$. The proof can be generalized to arbitrary $k \in \mathbb{N}$ to show that there exists a set S which is a weak interpolation set of order k for uniquely ergodic systems such that $d^*(S) > 1/k - \varepsilon$. However, from the proof of Theorem 1.6, there is no such a set that satisfies $d^*(S) > 1/k$. These facts lead to the following question.

Question 8.6. *For $k \in \mathbb{N}$, is there a set S which is a weak interpolation set of order k for*

uniquely ergodic systems such that $d^*(S) = 1/k$?

Among the classes of systems considered in this paper, for minimal systems, totally minimal systems, systems of zero entropy, and systems of finite entropy, weak interpolation of order 2 is equivalent to weak interpolation of all orders. On the other hand, for uniquely ergodic systems and strictly ergodic systems, these two notions are not equivalent. Our next question explores this equivalence for the class of totally transitive systems. More precisely, Theorem 1.3 implies that a weak interpolation set of all orders for totally transitive systems cannot be syndetic. Nevertheless, we do not know whether a weak interpolation set of order 2 for this class can be syndetic or not.

Question 8.7. *Does there exist a syndetic set which is weak interpolation of order 2 for the class of totally transitive systems?*

Our last question is concerned with the necessity of the hypothesis of Proposition 1.7. Recall that Proposition 1.7 shows that a syndetic set S satisfying the orbit closure of 1_S under the left shift has zero entropy cannot be weak interpolation of order 2 for uniquely ergodic systems. We do not know whether the entropy condition is necessary and so the following question is open:

Question 8.8. *Does there exist a syndetic set which is weak interpolation of order 2 for the class of uniquely ergodic systems?*

References

[1] E. Akin and J. Carlson. Conceptions of topological transitivity. *Topology Appl.*, 159(12):2815–2830, 2012.

[2] J. Briët and B. Green. Multiple correlation sequences not approximable by nilsequences. *Ergodic Theory Dynam. Systems*, 42(9):2711–2722, 2022.

[3] S. Drury. Sur les ensembles de Sidon. *C. R. Acad. Sci. Paris Sér. A-B*, 271:A162–A163, 1970.

[4] T. Eisner. A polynomial version of Sarnak's conjecture. *C. R. Math. Acad. Sci. Paris*, 353(7):569–572, 2015.

[5] S. Ferenczi, J. Kułaga-Przymus, and M. Lemańczyk. Sarnak's conjecture: what's new. In *Ergodic theory and dynamical systems in their interactions with arithmetics and combinatorics*, volume 2213 of *Lecture Notes in Math.*, pages 163–235. Springer, Cham, 2018.

[6] G. Fuhrmann and D. Kwietniak. On tameness of almost automorphic dynamical systems for general groups. *Bull. Lond. Math. Soc.*, 52(1):24–42, 2020.

[7] H. Furstenberg. Disjointness in ergodic theory, minimal sets, and a problem in Diophantine approximation. *Math. Systems Theory*, 1:1–49, 1967.

[8] H. Furstenberg. Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions. *J. d'Analyse Math.*, 31:204–256, 1977.

[9] H. Furstenberg. *Recurrence in ergodic theory and combinatorial number theory.* Princeton University Press, Princeton, N.J., 1981.

[10] E. Glasner. Divisible properties and the Stone-Čech compactification. *Canadian J. Math.*, 32(4):993–1007, 1980.

[11] E. Glasner, T. Tsankov, B. Weiss, and A. Zucker. Bernoulli disjointness. *Duke Math. J.*, 170(4):615–651, 2021.

[12] S. Glasner and B. Weiss. Interpolation sets for subalgebras of $l^\infty(\mathbf{Z})$. *Israel J. Math.*, 44(4):345–360, 1983.

[13] J. Griesmer. Special cases and equivalent forms of Katznelson's problem on recurrence. *Monatsh. Math.*, 200(1):63–79, 2023.

[14] D. Grow. A class of I_0-sets. *Collog. Math.*, 53(1):111–124, 1987.

[15] F. Hahn and Y. Katznelson. On the entropy of uniquely ergodic transformations. *Trans. Amer. Math. Soc.*, 126:335–360, 1967.

[16] S. Hartman. On interpolation by almost periodic functions. *Colloq. Math.*, 8:99–101, 1961.

[17] S. Hartman and C. Ryll-Nardzewski. Almost periodic extensions of functions. *Colloq. Math.*, 12:23–39, 1964.

[18] B. Host, B. Kra, and A. Maass. Variations on topological recurrence. *Monatsh. Math.*, 179(1):57–89, 2016.

[19] W. Huang. Tame systems and scrambled pairs under an abelian group action. *Ergodic Theory Dynam. Systems*, 26(5):1549–1567, 2006.

[20] A. Kanigowski, M. Lemańczyk, and M. Radziwiłł. Prime number theorem for analytic skew products. To appear in Ann. of Math. arXiv:2004.01125.

[21] D. Kerr and H. Li. Dynamical entropy in Banach spaces. *Invent. Math.*, 162(3):649–686, 2005.

[22] J. Kułaga-Przymus and M. Lemańczyk. Sarnak's conjecture from the ergodic theory point of view. In *Ergodic theory*, Encycl. Complex. Syst. Sci., pages 293–311. Springer, New York, 2023.

[23] K. Kunen and W. Rudin. Lacunarity and the Bohr topology. *Math. Proc. Cambridge Philos. Soc.*, 126(1):117–137, 1999.

[24] A. Le. Interpolation sets and nilsequences. *Colloq. Math.*, 162(2):181–199, 2020.

[25] A. Le. Sublacunary sets and interpolation sets for nilsequences. *Discrete Contin. Dyn. Syst.*, 42(4):1855–1871, 2022.

[26] Z. Lian and R. Shi. A counter-example for polynomial version of Sarnak's conjecture. *Adv. Math.*, 384:Paper No. 107765, 14, 2021.

[27] J.-F. Méla. Approximation diophantienne et ensembles lacunaires. *Bull. Soc. Math. France*, Mem. 19:26–54, 1969.

[28] R. Pavlov. Minimal zero entropy subshifts can be unrestricted along any sparse set. arXiv:2308.08013.

[29] R. Pavlov. Some counterexamples in topological dynamics. *Ergodic Theory Dynam. Systems*, 28(4):1291–1322, 2008.

[30] L. T. Ramsey. Fourier-Stieltjes transforms of measures with a certain continuity property. *J. Functional Analysis*, 25(3):306–316, 1977.

[31] C. Ryll-Nardzewski. Concerning almost periodic extensions of functions. *Colloq. Math.*, 12:235–237, 1964.

[32] P. Sarnak. Mobius randomness and dynamics. *Not. S. Afr. Math. Soc.*, 43(2):89–97, 2012.

[33] S. Sidon. Ein Satz über die absolute Konvergenz von Fourierreihen, in denen sehr viele Glieder fehlen. *Math. Ann.*, 96(1):418–419, 1927.

[34] S. Sidon. Verallgemeinerung eines Satzes über die absolute Konvergenz von Fourierreihen mit Lücken. *Math. Ann.*, 97(1):675–676, 1927.

[35] E. Strzelecki. On a problem of interpolation by periodic and almost periodic functions. *Colloq. Math.*, 11:91–99, 1963.

[36] B. Weiss. *Single orbit dynamics*, volume 95 of *CBMS Regional Conference Series in Mathematics*. American Mathematical Society, Providence, RI, 2000.

[37] Y. Zhang. Bounded gaps between primes. *Ann. of Math. (2)*, 179(3):1121–1174, 2014.

[38] R. Zimmer. Ergodic actions with generalized discrete spectrum. *Illinois J. Math.*, 20(4):555–588, 1976.

[39] R. Zimmer. Extensions of ergodic group actions. *Illinois J. Math.*, 20(3):373–409, 1976.

Andreas Koutsogiannis
Aristotle University of Thessaloniki
akoutsogiannis@math.auth.gr

Anh N. Le
University of Denver
anh.n.le@du.edu

Joel Moreira
University of Warwick
joel.moreira@warwick.ac.uk

Ronnie Pavlov
University of Denver
rpavlov@du.edu

Florian K. Richter
École Polytechnique Fédérale de Lausanne (EPFL)
f.richter@epfl.ch