INTEGER PART POLYNOMIAL CORRELATION SEQUENCES

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Abstract. Following an approach presented by N. Frantzikinakis in [7], we prove that any multiple correlation sequence defined by invertible measure preserving actions of commuting transformations with integer part polynomial iterates is the sum of a nilsequence and an error term, which is small in uniform density. As an intermediate result, we show that multiple ergodic averages with iterates given by the integer part of real valued polynomials converge in the mean. Also, we show that under certain assumptions the limit is zero. A transference principle, communicated to us by M. Wierdl, plays an important role in our arguments by allowing us to deduce results for $\mathbb{Z}$-actions from results for flows.

1. Introduction and main results

In this paper we study the structure of sequences of the form

$$(1)\qquad a(n)=\int f_0\cdot\left(\prod_{i=1}^{\ell}T_i^{[p_{i,1}(n)]}\right)f_1\cdot\ldots\cdot\left(\prod_{i=1}^{\ell}T_i^{[p_{i,m}(n)]}\right)f_m\,d\mu,\quad n\in\mathbb{N},$$

which we call *integer part polynomial correlation sequences*, where $\mathbb{N}$ is the set of positive integers, $[\cdot]$ denotes the integer part function, $T_1,\ldots,T_\ell\colon X\to X$ are invertible commuting measure preserving transformations on a probability space $(X,\mathcal{X},\mu)$, $f_0,f_1,\ldots,f_m\in L^\infty(\mu)$ are bounded functions and $p_{i,j}\in\mathbb{R}[t]$ are real valued polynomials for all $1\leq i\leq\ell$, $1\leq j\leq m$. For a measurable function g and transformations $S_1,\ldots,S_\ell: X\to X$, we denote by Sg the composition $g\circ S$ and by $\prod_{i=1}^{\ell}S_i$ the composition $S_1\circ\cdots\circ S_\ell$.

Throughout the article, we assume the functions f_i are bounded: $\|f_i\|_\infty\leq 1$.

Definition. For $\ell\in\mathbb{N}$, we call the tuple $(X,\mathcal{X},\mu,T_1,\ldots,T_\ell)$ a *system*, where $(X,\mathcal{X},\mu)$ is a probability space and $T_1,\ldots,T_\ell\colon X\to X$ are invertible commuting measure preserving transformations of X.

Any sequence of the form (1) is a special case of a multiple correlation sequence, i.e., sequence of the form

$$(2)\qquad \int f_0\cdot T_1^{n_1}f_1\cdot\ldots\cdot T_\ell^{n_\ell}f_\ell\,d\mu,$$

with $n_1,\ldots,n_\ell\in\mathbb{Z}$. The structure and the limiting behaviour of averages of multiple correlation sequences is a central topic in ergodic (Ramsey) theory. Although determining

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the precise structure of such sequences is an important open problem in the area, much partial progress has been made in recent years.

In order to state some relevant results, we recall the notion of an ℓ-step nilsequence.

Definition ([2])**.** For $\ell \in \mathbb{N}$, an *ℓ-step nilsequence* is a sequence of the form $(F(g^n\Gamma))$, where $F \in C(X)$, $X = G/\Gamma$, G is an ℓ-step nilpotent Lie group, Γ is a discrete cocompact subgroup and $g \in G$. A *0-step nilsequence* is a constant sequence.

In the special case of a single ergodic transformation (i.e., all the invariant sets under the transformation have measure either 0 or 1) where $T_i = T^i$, $i = 1, \ldots, \ell$, Bergelson, Host and Kra showed the following result:

Theorem ([2, Theorem 1.9])**.** *For $\ell \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T)$ be an ergodic system and $f_0, f_1, \ldots, f_\ell \in L^\infty(\mu)$. Then we have the decomposition*

$$\int f_0 \cdot T^n f_1 \cdot \ldots \cdot T^{\ell n} f_\ell \, d\mu = \mathcal{N}(n) + e(n), \quad n \in \mathbb{N},$$

where

(i) $(\mathcal{N}(n))$ *is a uniform limit of ℓ-step nilsequences with* $\|\mathcal{N}\|_\infty \leq 1$;

(ii) $\displaystyle\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |e(n)|^2 = 0.$

The polynomial iterates version of this result is due to Leibman ([11]). Also, the result of Bergelson, Host and Kra is proved in [12] without the ergodicity assumption.

All these results depend on the theory of characteristic factors, a tool that in a more general setting, say for correlation sequences involving actions of commuting transformations, proved to be extremely complex. Quite recently, Frantzikinakis (in [7]) showed, avoiding the use of the theory of characteristic factors, that modulo error terms, which are small in uniform density, correlation sequences of actions of commuting transformations are nilsequences ([7, Theorem 1.3]). More specifically, using the convergence result of Walsh from [14] and tools from [9] and [10], he proved:

Theorem ([7, Theorem 1.2])**.** *Let $\ell, m \in \mathbb{N}$ and $p_{i,j} \in \mathbb{Z}[t]$, $1 \leq i \leq \ell, 1 \leq j \leq m$, be polynomials. Then there exists $k \in \mathbb{N}$, $k = k(\ell, m, \max\deg(p_{i,j}))$, such that for every system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$, functions $f_0, f_1, \ldots, f_m \in L^\infty(\mu)$ and $\varepsilon > 0$, we have*

$$\int f_0 \cdot \left(\prod_{i=1}^{\ell} T_i^{p_{i,1}(n)}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{p_{i,m}(n)}\right) f_m \, d\mu = \mathcal{N}(n) + e(n), \quad n \in \mathbb{N}, \tag{3}$$

where

(i) $(\mathcal{N}(n))$ *is a k-step nilsequence with* $\|\mathcal{N}\|_\infty \leq 1$;

(ii) $\displaystyle\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |e(n)|^2 \leq \varepsilon.$

Our main result in this article is the following theorem.

Theorem 1.1. *Let $\ell, m \in \mathbb{N}$ and $p_{i,j} \in \mathbb{R}[t]$, $1 \leq i \leq \ell, 1 \leq j \leq m$, be polynomials. Then there exists $k \in \mathbb{N}$, $k = k(\ell, m, \max\deg(p_{i,j}))$, such that for every system $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$, functions $f_0, f_1, \dots, f_m \in L^\infty(\mu)$ and $\varepsilon > 0$, we have*

$$\int f_0 \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,1}(n)]}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,m}(n)]}\right) f_m \, d\mu = \mathcal{N}(n) + e(n), \quad n \in \mathbb{N}, \tag{4}$$

where

(i) *$(\mathcal{N}(n))$ is a k-step nilsequence with $\|\mathcal{N}\|_\infty \leq 1$;*

(ii) $\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |e(n)|^2 \leq \varepsilon.$

In the special case where $p_{i,j}$ are linear, so $p_{i,j}(n) = a_{i,j}n$, $a_{i,j} \in \mathbb{R}$, we can choose $k = m$.

For an application of Theorem 1.1, we consider the following subsets of $\ell^\infty(\mathbb{N})$ for $k \in \mathbb{N}$:

$$\mathcal{A}_k := \Big\{(\psi(n))\colon \psi \text{ is a } k\text{-step nilsequence}\Big\};$$

$$\mathcal{B}_k := \Big\{ \int f_0 \cdot \prod_{i=1}^{k} T^{(\ell_i - \ell_{k+1})n} f_i \, d\mu \colon (X, \mathcal{X}, \mu, T) \text{ is a system}, f_i \in L^\infty(\mu) \text{ and } \ell_i = \frac{(k+1)!}{i}\Big\};$$

$$\mathcal{C}_k := \Big\{ \int f_0 \cdot \left(\prod_{i=1}^{k} T_i^{[a_{i,1}n]}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{k} T_i^{[a_{i,k}n]}\right) f_k \, d\mu \colon (X, \mathcal{X}, \mu, T_1, \dots, T_k) \text{ is a system},$$
$$a_{1,1}, \dots, a_{k,1}, a_{1,2}, \dots, a_{1,k}, \dots, a_{k,k} \in \mathbb{R} \text{ and } f_i \in L^\infty(\mu)\Big\}.$$

It is proven in [7] that the sets $\mathcal{A}_k$ and $\mathcal{B}_k$ are linear subspaces of $\ell^\infty(\mathbb{N})$. By the same reasoning, we have that $\mathcal{C}_k$ is a linear subspace of $\ell^\infty(\mathbb{N})$ as well. Also, from Theorem 1.4 in [7], we have that, modulo sequences small in uniform density, the two subspaces $\mathcal{A}_k$ and $\mathcal{B}_k$ coincide. Namely, we have that

$$\overline{\mathcal{A}_k} = \overline{\mathcal{B}_k},$$

where the closure is taken in the topology of the seminorm $\|\cdot\|_2$ defined on $\ell^\infty(\mathbb{N})$ by

$$\|a\|_2^2 := \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |a(n)|^2. \tag{5}$$

For bounded sequences $(b(n))$, we denote by $\limsup_{N-M\to\infty} \left|\frac{1}{N-M} \sum_{n=M}^{N-1} b(n)\right|$ the limit $\lim_{N\to\infty} \sup_{M\in\mathbb{N}} \left|\frac{1}{N} \sum_{n=M}^{M+N-1} b(n)\right|$; this limit exists by subadditivity.

Using the linear case of Theorem 1.1, we will prove the following:

Theorem 1.2. *For every $k \in \mathbb{N}$ we have*

$$\overline{\mathcal{A}_k} = \overline{\mathcal{B}_k} = \overline{\mathcal{C}_k}.$$

As an intermediate step in the proof of Theorem 1.1, we also prove the following:

Theorem 1.3. *For $\ell, m \in \mathbb{N}$ let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $p_{i,j} \in \mathbb{R}[t]$ polynomials, $1 \leq i \leq \ell$, $1 \leq j \leq m$ and $f_1, \ldots, f_m \in L^\infty(\mu)$. Then the averages*

$$\frac{1}{N-M}\sum_{n=M}^{N-1}\left(\prod_{i=1}^{\ell} T_i^{[p_{i,1}(n)]}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,m}(n)]}\right) f_m \tag{6}$$

converge in $L^2(\mu)$ as $N - M \to \infty$.

In some special cases (see Theorems 6.2 and 6.3), we show convergence of (6) to 0.

The proof of Theorems 1.1 and 1.3 relies on the analogous results for flows. The technique employed below was first used in [5] and [13] in order to prove that when a sequence of real positive numbers is good for the single term pointwise (ergodic) convergence, then the respective sequence of its integer parts is also good. In these two surveys almost identical results were achieved independently and simultaneously. This method was later adapted by M. Wierdl (in [15]) to deal with multiple term averages (see Theorem 3.2).

Theorems 1.1 and 1.3 are results in a more general scheme and can be considered as the first step towards the understanding of the structure of correlation sequences with iterates given by generalized polynomials.

Definition ([4])**.** We denote by $\mathcal{G}$ the smallest family of $\mathbb{N} \to \mathbb{Z}$ functions containing $\mathbb{Z}[n]$ that forms an algebra under addition and multiplication and having the property that for every $f_1, \ldots, f_r \in \mathcal{G}$ and $c_1, \ldots, c_r \in \mathbb{R}$, $\left[\sum_{i=1}^{r} c_i f_i\right] \in \mathcal{G}$. The members of $\mathcal{G}$ are called *generalized polynomials.*

Conjecture. Theorems 1.1 and 1.3 hold if $p_{i,j}$ are generalized polynomials.

Note that for Theorem 1.1, the conjecture in its generality is completely open. As for Theorem 1.3, only the one-term case is known and is due to Bergelson and Leibman ([3]). Even the case where all the transformations are equal is not known.

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2. Definitions and Main Ideas

2.1. **Anti-uniformity and regularity.** For the convenience of the reader, in this subsection, we repeat the necessary material from Section 2 from [7].

First we recall the notion of uniformity seminorms (a slight variant of the uniformity seminorms defined by B. Host and B. Kra in [10]).

Definition ([7])**.** Let $k \in \mathbb{N}$ and $a \colon \mathbb{N} \to \mathbb{C}$ be a bounded sequence.

(i) Given a sequence of intervals $\mathbf{I} = (I_N)$ with lengths tending to infinity, we say that the sequence $(a(n))$ is *distributed regularly along* $\mathbf{I}$ if the limit

$$\lim_{N\to\infty} \frac{1}{|I_N|} \sum_{n\in I_N} a_1(n+h_1) \cdot \ldots \cdot a_r(n+h_r)$$

exists for every $r \in \mathbb{N}$ and $h_1, \ldots, h_r \in \mathbb{N}$, where a_i is either a or $\bar{a}$.

(ii) If $\mathbf{I}$ is as in (i) and $(a(n))$ is distributed regularly along $\mathbf{I}$, we define inductively

$$\|a\|_{\mathbf{I},1} := \lim_{N\to\infty} \Big| \frac{1}{|I_N|} \sum_{n\in I_N} a(n) \Big|;$$

and for $k \geq 2$ (one can show as in [10, Proposition 4.3] that the next limit exists)

$$\|a\|_{\mathbf{I},k}^{2^k} := \lim_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \|\sigma_h a \cdot \bar{a}\|_{\mathbf{I},k-1}^{2^{k-1}},$$

where σ_h is the shift transformation defined by $(\sigma_h a)(n) := a(n+h)$.

(iii) If $(a(n))$ is a bounded sequence we let

$$\|a\|_{U_k(\mathbb{N})} := \sup_{\mathbf{I}} \|a\|_{\mathbf{I},k},$$

where the sup is taken over all sequences of intervals $\mathbf{I}$ with lengths tending to infinity along which the sequence $(a(n))$ is distributed regularly.

Definition ([7])**.** Let $k \in \mathbb{N}$. We say that the bounded sequence $a\colon \mathbb{N} \to \mathbb{C}$ is

(i) *k-anti-uniform* if there exists $C := C(k, a)$ such that

$$\limsup_{N-M\to\infty} \Big| \frac{1}{N-M} \sum_{n=M}^{N-1} a(n)b(n) \Big| \leq C\, \|b\|_{U_k(\mathbb{N})}$$

for every $b \in \ell^\infty(\mathbb{N})$.

(ii) *k-regular* if the limit

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} a(n)\psi(n)$$

exists for every $(k-1)$-step nilsequence $(\psi(n))$.

It turns out that k-anti-uniformity and k-regularity give a sufficient condition on the required decomposition of sequences of the form (3) ([7, Theorem 1.3]). Our goal is to prove that sequences of the form (1) are k-regular for all k and k-weak-anti-uniform for some k (see Theorem 2.1- we attribute its proof in [7] and we present here a sketch of it as it has the same arguments with [7, Theorem 1.3]).

Definition. Let $k \in \mathbb{N}$. We say that the bounded sequence $a : \mathbb{N} \to \mathbb{C}$ is *k-weak-anti-uniform* if for every $0 < \delta < 1$ there exists a positive constant $C_\delta := C_\delta(k, a)$ and a term c_δ, with $c_\delta \to 0$ as $\delta \to 0^+$, such that for every $b \in \ell^\infty(\mathbb{N})$,

$$\limsup_{N-M\to\infty} \Big| \frac{1}{N-M} \sum_{n=M}^{N-1} a(n)b(n) \Big| \leq C_\delta\, \|b\|_{U_k(\mathbb{N})} + c_\delta.$$

Theorem 2.1 ([7, Theorem 1.3]). *Let $k \in \mathbb{N}$ and $a\colon \mathbb{N} \to \mathbb{C}$ be a sequence with $\|a\|_\infty \leq 1$ that is k-weak-anti-uniform and k-regular. Then, for every $\varepsilon > 0$, we have the decomposition*

$$a(n) = \mathcal{N}(n) + e(n), \quad n \in \mathbb{N},$$

where

(i) $(\mathcal{N}(n))$ *is a* $(k-1)$*-step nilsequence with* $\|\mathcal{N}\|_\infty \leq 1$*;*

(ii) $\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |e(n)|^2 \leq \varepsilon$.

Proof. ([7, Theorem 1.3]) We first remark that the limit

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} |a(n)|^2$$

exists. Let $Y := \Big\{(\psi(n))\colon \psi \text{ is a } (k-1)\text{-step nilsequence}\Big\}$ and $X := \text{span}\{Y, a\}$.

On $X \times X$ we define the bilinear form

$$\langle f, g\rangle := \lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} f(n)\overline{g}(n).$$

Since the limit exists for $f, g \in X$, this bilinear form induces the seminorm $\|f\|_2 := \sqrt{\langle f, f\rangle}$ (note that this is the restriction, on the space X, of the seminorm defined in (5)).

Let $\varepsilon > 0$. There exists $\delta_0 > 0$ such that for any sequence $(b(n)) \in \ell^\infty(\mathbb{N})$ we have $\limsup_{N\to\infty} \Big|\frac{1}{|I_N|} \sum_{n\in I_N} a(n)b(n)\Big| \leq C_{\delta_0} \|b\|_{U_k(\mathbb{N})} + \varepsilon/4$. We can assume that $C_{\delta_0} \geq 1$.

For $d := \inf\{\|a - y\|_2 : y \in Y\}$ and $\delta := \Big(\varepsilon(8C_{\delta_0})^{-1}\Big)^{2^k}$, there exists $y_0 \in Y$ such that

$$\|a - y_0\|_2^2 \leq d^2 + \delta^2. \tag{7}$$

We can also assume that $\|y_0\|_\infty \leq 1$. Using (7) we can prove that

$$\sup_{y\in Y\colon \|y\|_2\leq 1} |\langle a - y_0, y\rangle| \leq 2\delta. \tag{8}$$

Since the set $\{y \in Y\colon \|y\|_2 \leq 1\}$ contains all the $(k-1)$-step nilsequences that are bounded by 1, we have that

$$\|a - y_0\|_{U_k(\mathbb{N})} \leq (2\delta)^{2^{-k}}. \tag{9}$$

Let $\mathcal{N} := y_0$ and $e := a - y_0$. Then $a = \mathcal{N} + e$, and $(\mathcal{N}(n))$ is an $(k-1)$-step nilsequence with $\|\mathcal{N}\|_\infty \leq 1$. Using the fact that $(a(n))$ is k-weak-anti-uniform and relation (9), we get

$$|\langle a, e\rangle| \leq C_{\delta_0} \|e\|_{U_k(\mathbb{N})} + \varepsilon/4 = C_{\delta_0} \|a - y_0\|_{U_k(\mathbb{N})} + \varepsilon/4 \leq \varepsilon/2.$$

Furthermore, from (8) we have

$$|\langle \mathcal{N}, e\rangle| \leq \varepsilon/2.$$

Combining the last two estimates we deduce that $\|e\|_2^2 = \langle e, e\rangle \leq |\langle a, e\rangle| + |\langle \mathcal{N}, e\rangle| \leq \varepsilon$. $\square$

Theorem 2.1 allows us to prove Theorem 1.1 by showing that a sequence of the form (1) is k-weak-anti-uniform and k-regular for some k; this is accomplished in Sections 3 and 4.

2.2. **The seminorms $\|\!|\cdot|\!\|_k$.** We follow [9] and [6] for the inductive definition of the seminorms $\|\!|\cdot|\!\|_k$. More specifically, the definition that we use here follows from [9] (in the ergodic case), [6] (in the general case) and the use of von Neumann's ergodic theorem.

Let $(X,\mathcal{X},\mu,T)$ be a system and $f\in L^\infty(\mu)$. We define inductively the seminorms $\|\!|f|\!\|_{k,\mu,T}$ as follows:

$$\|\!|f|\!\|_{1,\mu,T}:=\|\mathbb{E}(f|\mathcal{I})\|_{L^2(\mu)},$$

where $\mathcal{I}$ is the σ-algebra of T-invariant sets and $\mathbb{E}(f|\mathcal{I})$ is the conditional expectation of f with respect to $\mathcal{I}$, satisfying $\int \mathbb{E}(f|\mathcal{I})\,d\mu=\int f\,d\mu$ and $T\mathbb{E}(f|\mathcal{I})=\mathbb{E}(Tf|\mathcal{I})$.

For $k\geq 1$, we let

$$\|\!|f|\!\|_{k+1,\mu,T}^{2^{k+1}}:=\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}\|\!|\bar{f}\cdot T^nf|\!\|_{k,\mu,T}^{2^k}.$$

All these limits exist and define seminorms (see [9]). By using von Neumann's ergodic theorem, we get $\|\!|f|\!\|_{1,\mu,T}^2=\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M}^{N-1}\int \bar{f}\cdot T^nf\;d\mu$, and more generally, for every $k\geq 1$ we have that

$$\|\!|f|\!\|_{k,\mu,T}^{2^k}:=\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n_1=M}^{N-1}\cdots\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n_k=M}^{N-1}\int\prod_{\vec{\epsilon}\in\{0,1\}^k}\mathcal{C}^{|\vec{\epsilon}|}T^{\vec{\epsilon}\cdot\vec{n}}f\;d\mu, \tag{10}$$

where $\vec{\epsilon}=(\epsilon_1,\ldots,\epsilon_k)$, $\vec{n}=(n_1,\ldots,n_k)$, $|\vec{\epsilon}|=\epsilon_1+\ldots+\epsilon_k$, $\vec{\epsilon}\cdot\vec{n}=\epsilon_1n_1+\ldots+\epsilon_kn_k$ and for $z\in\mathbb{C}$, $k\in\mathbb{N}\cup\{0\}$ we let $\mathcal{C}^k(z)=\begin{cases} z & \text{if } k \text{ is even}\\ \bar{z} & \text{if } k \text{ is odd}\end{cases}$.

We also remark that $\|\!|f\otimes\bar{f}|\!\|_{k,\mu\times\mu,T\times T}\leq\|\!|f|\!\|_{k+1,\mu,T}^2$ for all $k\in\mathbb{N}$, which follows from (10) and the ergodic theorem, $\|\!|f|\!\|_{k,\mu,T}\leq\|\!|f|\!\|_{k+1,\mu,T}$ for all $k\in\mathbb{N}$ (by using Lemma 3.9 from [9]) and $\|\!|f|\!\|_{k,\mu,T}=\|\!|f|\!\|_{k,\mu,T^{-1}}$ for all $k\in\mathbb{N}$, which follows from (10).

It is a deep fact, shown in [9], that for ergodic systems we have $\|\!|f|\!\|_{k+1}=0$ if and only if f is orthogonal to the largest k-step "nil-factor" of the system. We will not use this fact here though.

In order to recall a convergence result from [6], we need the following notion:

Definition ([6])**.** Let $\ell,m\in\mathbb{N}$. Given $(p_{i,j})_{1\leq i\leq\ell,1\leq j\leq m}\subseteq\mathbb{Z}[t]$ (resp. $\mathbb{R}[t]$) we define the *ordered family of m polynomial ℓ-tuples*: $(\mathcal{P}_1,\ldots,\mathcal{P}_\ell)_m:=((p_{1,1},\ldots,p_{\ell,1}),\ldots,(p_{1,m},\ldots,p_{\ell,m}))$. This family is *nice in* $\mathbb{Z}[t]$ (resp. $\mathbb{R}[t]$) if

(i) $\deg(p_{1,1})\geq\deg(p_{1,j})$ for $1\leq j\leq m$;
(ii) $\deg(p_{1,1})>\deg(p_{i,j})$ for $2\leq i\leq\ell$, $1\leq j\leq m$; and
(iii) $\deg(p_{1,1}-p_{1,j})>\deg(p_{i,1}-p_{i,j})$ for $2\leq i\leq\ell$, $2\leq j\leq m$.

A nice family of polynomials with maximum degree 1 has only one non-zero term.

Using the theory of characteristic factors, Chu, Frantzikinakis and Host showed:

Theorem 2.2 ([6, Theorem 1.2])**.** *For $\ell \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $p_1, \ldots, p_\ell \in \mathbb{Z}[t]$ be non-constant polynomials with distinct degrees and maximum degree d, and $f_1, \ldots, f_\ell \in L^\infty(\mu)$. There exists $k = k(\ell, d) \in \mathbb{N}$ such that if $\|\!|f_i|\!\|_{k,\mu,T_i} = 0$ for some $1 \leq i \leq \ell$, then*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} T_1^{p_1(n)} f_1 \cdot \ldots \cdot T_\ell^{p_\ell(n)} f_\ell \right\|_{L^2(\mu)} = 0.$$

Also, under a stronger hypothesis, the same authors proved:

Proposition 2.3 ([6, Proposition 5.1])**.** *For $\ell, m \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)_m$ be a nice family of ℓ-tuples in $\mathbb{Z}[t]$ with maximum degree d, and $f_1, \ldots, f_m \in L^\infty(\mu)$. There exists $k = k(d, \ell, m) \in \mathbb{N}$ such that if $\|\!|f_1|\!\|_{k,\mu,T_1} = 0$, then*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{\ell} T_i^{p_{i,1}(n)} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{p_{i,m}(n)} \right) f_m \right\|_{L^2(\mu)} = 0.$$

Remark. In these two last results from [6], k can be chosen arbitrarily large.

3. Regularity

In this section we will prove that a sequence of the form in (1) is k-regular for all $k \in \mathbb{N}$.

Proposition 3.1 ([7, Proposition 2.4])**.** *For $k \in \mathbb{N}$, let $(\psi(n))$ be a $(k-1)$-step nilsequence. For every $\varepsilon > 0$ there exists a system $(X, \mathcal{X}, \nu, S)$ and functions $f_1, \ldots, f_k \in L^\infty(\nu)$, such that the sequence $(b(n))$ defined by*

$$b(n) := \int S^{\ell_1 n} f_1 \cdot \ldots \cdot S^{\ell_k n} f_k \; d\nu, \quad n \in \mathbb{N}, \tag{11}$$

where $\ell_i := k!/i$ for $i = 1, \ldots, k$, satisfies

$$\|\psi - b\|_\infty \leq \varepsilon.$$

In order to obtain the k-regularity for any $k \in \mathbb{N}$ of a sequence $(a(n))$ as in (1), we have to check that the limit $\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} a(n)\psi(n)$ exists for every $(k-1)$-step nilsequence $(\psi(n))$. By Proposition 3.1, it suffices to check that the limit

$$\lim_{N-M\to\infty} \frac{1}{N-M} \sum_{n=M}^{N-1} a(n)b(n) \tag{12}$$

exists for every sequence $(b(n))$ of the form $\int S^{\ell_1 n} g_1 \cdot \ldots \cdot S^{\ell_k n} g_k \; d\nu$, where $\ell_1, \ldots, \ell_k \in \mathbb{N}$, $(Y, \mathcal{Y}, \nu, S)$ is a system, and $g_1, \ldots, g_k \in L^\infty(\nu)$. To verify that this limit exists, we will adapt a trick of M. Wierdl - originally for $\mathbb{R}^2$ measure preserving flows (see definition below) and Cesaro averages - in our setting.

Definition. Let $S \subseteq \mathbb{N}$. We define the *upper Banach density* of S, $d^*(S)$, to be the number

$$d^*(S) = \limsup_{N-M\to\infty} \frac{|S \cap [M, \ldots, N)|}{N-M}.$$

Definition. Let $r \in \mathbb{N}$ and $(X, \mathcal{X}, \mu)$ be a probability space. We call a family $(T_t)_{t\in\mathbb{R}^r}$ of measure preserving transformations $T_t\colon X \to X$ a *measure preserving flow* if it satisfies

$$T_{s+t} = T_s \circ T_t$$

for all $s, t \in \mathbb{R}^r$.

The following result contains the central idea of passing from results for flows to results for $\mathbb{Z}$-actions. Essential for this is the (ℓm)-dimensional variant of the special flow above a system under the constant ceiling function 1 first defined (for $\ell = m = 1$) in [5] and [13].

Theorem 3.2 ([15]). *Let $\ell, m \in \mathbb{N}$. Suppose that the sequences of real numbers $(a_{i,j}(n))$, $1 \le i \le \ell$, $1 \le j \le m$, satisfy the following two properties:*

(i) *for any $\mathbb{R}^{\ell m}$ measure preserving flow $\prod_{i=1}^{\ell} T_{i,a_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,a_{i,m}}$, where the $T_{i,\cdot}$ are flows on the probability space $(X, \mathcal{X}, \mu)$ and $f_1, \ldots, f_m \in L^\infty(\mu)$, the averages*

$$\frac{1}{|I_N|}\sum_{n\in I_N}\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot a_{i,1}(n)}\right) f_1 \cdot \ldots \cdot \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot a_{i,m}(n)}\right) f_m$$

converge in $L^2(\mu)$ as $N \to \infty$ ($|I_N| \to \infty$ as $N \to \infty$), where δ_{ij} is the Kronecker delta function with $\delta_{ij} = 1$ if $i = j$ and 0 elsewhere and $T_{i,0}$ the identity for all i;

(ii) $\lim_{\delta\to 0^+} d^*\left(\left\{n\colon \{a_{i,j}(n)\} \in [1-\delta, 1)\right\}\right) = 0$ *for all $1 \le i \le \ell$, $1 \le j \le m$, where $\{\cdot\}$ denotes the fractional part.*

Then the averages

$$\frac{1}{|I_N|}\sum_{n\in I_N}\left(\prod_{i=1}^{\ell} T_i^{[a_{i,1}(n)]}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,m}(n)]}\right) f_m$$

also converge in $L^2(\mu)$, as $N \to \infty$, for any system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and functions $f_1, \ldots, f_m \in L^\infty(\mu)$.

Proof (We use the notation and arguments of Wierdl from [15]*).* For the given transformations on X, we define the $\mathbb{R}^{\ell m}$-action $\prod_{i=1}^{\ell} T_{i,a_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,a_{i,m}}$ on the probability space $Y = X \times [0,1)^{\ell m}$, with the measure $\nu = \mu \times \lambda^{\ell m}$ (λ is the Lebesgue measure on $[0,1)$), by

$$\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,a_{i,j}}(x, b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,2}, \ldots, b_{1,m}, \ldots, b_{\ell,m}) =$$

$$\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_i^{[a_{i,j}+b_{i,j}]}x, \{a_{1,1}+b_{1,1}\}, \ldots, \{a_{\ell,1}+b_{\ell,1}\}, \ldots, \{a_{1,m}+b_{1,m}\}, \ldots, \{a_{\ell,m}+b_{\ell,m}\}\right).$$

Since $T_1, \ldots, T_\ell$ are commuting measure preserving transformations, and since $[x + \{y\}] + [y] = [x + y]$, it is easy to check that the above action defines a measure preserving flow on the product probability space Y. Note that this is nothing but the (ℓm)-dimensional variant of the "suspension" flow above a system under the constant ceiling function 1.

For a bounded function f on X, we define its extension $\hat{f}$ on Y by

$$\hat{f}(x, b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,2}, \ldots, b_{1,m}, \ldots, b_{\ell,m}) = f(x).$$

Note that if $a_{i,j}$, for $1 \leq i \leq \ell$, $1 \leq j \leq m$, are real numbers, then

$$\prod_{k=1}^{m}\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jk}\cdot a_{i,k}}\right)\hat{f}_k(x,0,\ldots,0) = \left(\prod_{i=1}^{\ell} T_i^{[a_{i,1}]}\right) f_1(x)\cdot\ldots\cdot\left(\prod_{i=1}^{\ell} T_i^{[a_{i,m}]}\right) f_m(x).$$

We want to show that the averages

$$\frac{1}{|I_N|}\sum_{n\in I_N}\prod_{k=1}^{m}\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jk}\cdot a_{i,k}(n)}\right)\hat{f}_k(x,0,\ldots,0)$$

converge in $L^2(X,0,\ldots,0)$ as $N\to\infty$. Let

$$A_N(x,b_{1,1},\ldots,b_{\ell,m}) = \frac{1}{|I_N|}\sum_{n\in I_N}\prod_{k=1}^{m}\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jk}\cdot a_{i,k}(n)}\right)\hat{f}_k(x,b_{1,1},\ldots,b_{\ell,m}),$$

and assume to the contrary that $(A_N(x,0,\ldots,0))$ is not Cauchy in $L^2(X,0,\ldots,0)$. This means that we can find a sequence (N_k) going to infinity, with

$$\int_{(X,0,\ldots,0)}\left|A_{N_{k+1}}(x,0,\ldots,0) - A_{N_k}(x,0,\ldots,0)\right|^2 d\mu > c, \quad k=1,2,\ldots, \tag{13}$$

for some positive number c. By the hypothesis, $(A_N(x,b_{1,1},\ldots,b_{\ell,m}))$ is Cauchy in $L^2(Y)$. By Fubini's theorem, for any given positive ε, we can find $(b_{1,1},\ldots,b_{\ell,m})$ arbitrarily close to $(0,\ldots,0)$ and $k\in\mathbb{N}$ so that

$$\int_{(X,b_{1,1},\ldots,b_{\ell,m})}\left|A_{N_{k+1}}(x,b_{1,1},\ldots,b_{\ell,m}) - A_{N_k}(x,b_{1,1},\ldots,b_{\ell,m})\right|^2 d\mu < \varepsilon.$$

We will show that for any given $\varepsilon > 0$, there exists $\delta > 0$ so that if $0 \leq b_{i,j} \leq \delta$ for $1\leq i\leq \ell$, $1\leq j\leq m$, then, for every $x\in X$ and large enough N, we have

$$\left|A_N(x,b_{1,1},\ldots,b_{\ell,m}) - A_N(x,0,\ldots,0)\right| < \varepsilon. \tag{14}$$

This will yield the desired contradiction with (13).

Let $0<\delta<1$ (we will choose it later), and assume that $0\leq b_{i,j}\leq\delta$ for all $1\leq i\leq\ell$, $1\leq j\leq m$. In (14) we have to compare terms of the form

$$\prod_{k=1}^{m}\hat{f}_k\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jk}\cdot a_{i,k}(n)}(x,b_{1,1},\ldots,b_{\ell,m})\right) \text{ with } \prod_{k=1}^{m}\hat{f}_k\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jk}\cdot a_{i,k}(n)}(x,0,\ldots,0)\right).$$

So, by the definition of the flow and the $\hat{f}_j$'s, we need to compare

$$\prod_{j=1}^{m} f_j\left(\prod_{i=1}^{\ell} T_i^{[a_{i,j}(n)+b_{i,j}]}x\right) \quad\text{with}\quad \prod_{j=1}^{m} f_j\left(\prod_{i=1}^{\ell} T_i^{[a_{i,j}(n)]}x\right).$$

Since $b_{i,j} \leq \delta$, if n is such that all $\{a_{i,j}(n)\} < 1-\delta$, then $T_i^{[a_{i,j}(n)+b_{i,j}]} = T_i^{[a_{i,j}(n)]}$. Therefore, it suffices to deal with those n's for which some $\{a_{i,j}(n)\} \geq 1-\delta$. By condition (ii), we have that the upper Banach density of these n's is as small as we want by taking δ small. All f_j are bounded, so the combined density of these terms in the averages $A_N(x, b_{1,1}, \dots, b_{\ell,m})$ and $A_N(x, 0, \dots, 0)$ is as small as we want independently of x. Hence, if δ is chosen sufficiently small, we get (14). □

Now, by using the previous result, we will prove Theorem 1.3.

Proof of Theorem 1.3. It suffices to show that for $a_{i,j} = p_{i,j}$, real valued polynomials, the conditions of Theorem 3.2 are satisfied. Using Walsh's convergence result for commuting measure preserving transformations from [14] we have condition (i). For example, if $p(t) = b_r t^r + \cdots + b_1 t + b_0 \in \mathbb{R}[t]$, we write $T_{p(n)} = (T_{b_r})^{n^r} \cdot \ldots \cdot (T_{b_1})^n \cdot T_{b_0}$ and we use Walsh's result for the commuting measure preserving transformations $S_1 = T_{b_1}, \dots, S_r = T_{b_r}$. Real valued polynomials also satisfy condition (ii). Indeed, let $p(t) = b_r t^r + \cdots + b_1 t + b_0 \in \mathbb{R}[t]$. If $b_i \notin \mathbb{Q}$ for some $1 \leq i \leq r$, then we have the condition from Weyl's equidistribution criterion, since $(p(n))$ is uniformly distributed (mod 1). If $b_i \in \mathbb{Q}$ for all $1 \leq i \leq r$, then the sequence $(p(n))$ is periodic (mod 1) and condition (ii) is immediate. □

In order to show k-regularity, it is sufficient to show that the limit (12) exists for every sequence $(b(n))$ of the form $\int S^{\ell_1 n} g_1 \cdot \ldots \cdot S^{\ell_r n} g_r \, d\nu$, where $r \in \mathbb{N}$ is arbitrary, $\ell_1, \dots, \ell_r \in \mathbb{N}$, $(Y, \mathcal{Y}, \nu, S)$ is a system, and $g_1, \dots, g_r \in L^\infty(\nu)$. To show that the averages of

$$\int f_0 \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,1}(n)]} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,m}(n)]} \right) f_m \, d\mu \cdot \int S^{\ell_1 n} g_1 \cdot \ldots \cdot S^{\ell_r n} g_r \, d\nu$$

converge, it suffices to show convergence in $L^2(\mu)$ of the averages of

$$\left(\prod_{i=1}^{\ell} T_i^{[p_{i,1}(n)]} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,m}(n)]} \right) f_m \cdot \int S^{\ell_1 n} g_1 \cdot \ldots \cdot S^{\ell_r n} g_r \, d\nu.$$

By Theorem 1.3 for the $\ell + r$ commuting measure preserving transformations $T_i \times \mathrm{id}$, $\mathrm{id} \times S^{\ell_j}$, $1 \leq i \leq \ell$, $1 \leq j \leq r$, acting on $X \times Y$ with the measure $\tilde{\mu} := \mu \times \nu$, and the functions $f_i \otimes 1$, $1 \otimes g_j$, $1 \leq i \leq \ell$, $1 \leq j \leq r$, we have that the averages of

$$\prod_{j=1}^{m} \left(\left(\prod_{i=1}^{\ell} (T_i \times \mathrm{id})^{[p_{i,j}(n)]} \right) (f_j \otimes 1) \right) \cdot \prod_{j=1}^{r} \left((\mathrm{id} \times S^{\ell_j})^n (1 \otimes g_j) \right)$$

converge in $L^2(\mu \times \nu)$. Integrating by $d\nu$, we get the required convergence.

4. Weak-Anti-Uniformity

In this section we show that a sequence of the form in (1) is k-weak-anti-uniform for some k.

Theorem 4.1. *Let $\ell, m \in \mathbb{N}$. Suppose that the sequences of real numbers $(a_{i,j}(n))$ satisfy the following two properties:*

(i) *for any* $\mathbb{R}^{\ell m}$ *measure preserving flow* $\prod_{i=1}^{\ell} T_{i,a_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,a_{i,m}}$, *where the* $T_{i,\cdot}$ *are flows on the probability space* $(X, \mathcal{X}, \mu)$ *and* $f_1, \ldots, f_m \in L^\infty(\mu)$, *the sequence*

$$\tilde{a}(n) = \int f_0 \cdot \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot a_{i,1}(n)}\right) f_1 \cdot \ldots \cdot \left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot a_{i,m}(n)}\right) f_m \, d\mu$$

is k*-anti-uniform for some* k *depending only on* ℓ, m *and* $a_{i,j}$;

(ii) $\lim_{\delta\to 0^+} d^*\left(\left\{n\colon \{a_{i,j}(n)\} \in [1-\delta, 1)\right\}\right) = 0$ *for all* $1 \le i \le \ell$, $1 \le j \le m$.

Then the sequence

$$a(n) = \int f_0 \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,1}(n)]}\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,m}(n)]}\right) f_m \, d\mu$$

is k*-weak-anti-uniform for every system* $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ *and* $f_0, f_1, \ldots, f_m \in L^\infty(\mu)$.

Proof. Let $0 < \delta < 1$. We define the same action $\mathbb{R}^{\ell m}$ on $Y = X \times [0,1)^{\ell m}$ as we did in the proof of Theorem 3.2. If $f_0, f_1, \ldots, f_m$ are bounded functions on X, for every $(b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,2}, \ldots, b_{1,m}, \ldots, b_{\ell,m}) \in [0,1)^{\ell m}$ we define the Y-extensions

$$\hat{f}_j(x, b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,2}, \ldots, b_{1,m}, \ldots, b_{\ell,m}) = f_j(x), \;\; 1 \le j \le m; \;\text{ and}$$

$$\hat{f}_0(x, b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,m}) = f_0(x) \cdot \mathbf{1}_{[0,\delta]^{\ell m}}(b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,m}).$$

Then, we have

$$\left|\delta^{\ell m} a(n) - \tilde{a}(n)\right| =$$

$$\Big|\int_{[0,\delta]^{\ell m}}\int_X f_0(x) \cdot \left(\prod_{j=1}^{m} f_j\left(\prod_{i=1}^{\ell} T_i^{[a_{i,j}(n)]}x\right) - \prod_{j=1}^{m} f_j\left(\prod_{i=1}^{\ell} T_i^{[a_{i,j}(n)+b_{i,j}]}x\right)\right) d\mu\, d\lambda^{\ell m}\Big|.$$

Since all $b_{i,j} \le \delta$ in the integrand, as in Theorem 3.2, we need only to consider the case when some $\{a_{i,j}(n)\} \ge 1-\delta$. For any $(b(n)) \in \ell^\infty(\mathbb{N})$ and $N > M$, we have

$$\Big|\frac{1}{N-M}\sum_{n=M}^{N-1} a(n)b(n)\Big| \le \frac{1}{\delta^{\ell m}}\Big|\frac{1}{N-M}\sum_{n=M}^{N-1} \tilde{a}(n)b(n)\Big| + c_\delta,$$

where $c_\delta \to 0$ as $\delta \to 0^+$. If C is the constant we get from the k-anti-uniformity of $\tilde{a}$, then

$$\limsup_{N-M\to\infty}\Big|\frac{1}{N-M}\sum_{n=M}^{N-1} a(n)b(n)\Big| \le \frac{C}{\delta^{\ell m}} \cdot \|b\|_{U_k(\mathbb{N})} + c_\delta,$$

yielding the result. □

Remark. As we showed in the proof of Theorem 1.3, if every $a_{i,j}$ is a polynomial $p_{i,j} \in \mathbb{R}[t]$, then condition (ii) of Theorem 4.1 is satisfied. In order to get condition (i), as described by Theorem 1.2 in [7] for the corresponding sequences, we have to successively make use of Lemma 6.1 (using the van der Corput operation, choosing every time appropriate polynomials in order to have reduction in complexity), stated in Section 6 below. In this case, k can be chosen to be equal to $d+1$, where d is the number of steps we need to reduce the polynomials to constant ones by using PET induction, introduced in [1]. This d, and so k as well, only depends on ℓ, m and the maximum degree of the polynomials $p_{i,j}$. For more information and details on the van der Corput operation and the scheme of PET induction we are using here, we refer the reader to [6].

This remark combined with Theorem 4.1 gives that every sequence $(a(n))$ of the form (1) is k-weak-anti-uniform for some positive integer $k = k(\ell, m, \max\deg(p_{i,j}))$.

5. Convergence and seminorm estimates

In this section we prove Theorem 5.1 and an implication of it (Proposition 5.4).

Theorem 5.1. *Let $\ell, m \in \mathbb{N}$. Suppose that the sequences of real numbers $(a_{i,j}(n))$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$, satisfy the following two properties:*

(i) *for any $\mathbb{R}^{\ell m}$ measure preserving flow $\prod_{i=1}^{\ell} T_{i,a_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,a_{i,m}}$, where the $T_{i,\cdot}$ are flows on the probability space $(X, \mathcal{X}, \mu)$ and $f_1, \ldots, f_m \in L^\infty(\mu)$, we have*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{j=1}^{m} \prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot a_{i,1}(n)} \right) f_1 \cdot \ldots \cdot \left(\prod_{j=1}^{m} \prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot a_{i,m}(n)} \right) f_m \right\|_{L^2(\mu)} = 0;$$

(ii) $\lim_{\delta\to 0^+} d^*\left(\left\{n \colon \{a_{i,j}(n)\} \in [1-\delta, 1)\right\}\right) = 0$ *for all $1 \leq i \leq \ell$, $1 \leq j \leq m$.*

Then, for any system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and functions $f_1, \ldots, f_m \in L^\infty(\mu)$, we have

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{\ell} T_i^{[a_{i,1}(n)]} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,m}(n)]} \right) f_m \right\|_{L^2(\mu)} = 0.$$

Proof. Let $0 < \delta < 1$. We define the same $\mathbb{R}^{\ell m}$-action on $Y = X \times [0,1)^{\ell m}$ and the Y-extensions $\hat{f}_j$ of f_j, $0 \leq j \leq m$, as in the proof of Theorems 3.2 and 4.1. Let

$$\tilde{a}(n) = \hat{f}_0 \cdot \left(\prod_{j=1}^{m} \prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot a_{i,1}(n)} \right) \hat{f}_1 \cdot \ldots \cdot \left(\prod_{j=1}^{m} \prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot a_{i,m}(n)} \right) \hat{f}_m,$$

for every $x \in X$ we define

$$a'(n)(x) = \int_{[0,1)^{\ell m}} \tilde{a}(n)(x, b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,m})\, d\lambda^{\ell m},$$

where the integration is with respect to the variables $b_{i,j}$.

Letting $a(n) := \left(\prod_{i=1}^{\ell} T_i^{[a_{i,1}(n)]} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,m}(n)]} \right) f_m$, for every $x \in X$, we have that

$$\left|\delta^{\ell m} a(n)(x) - a'(n)(x)\right| =$$

$$\Big| \int_{[0,\delta]^{\ell m}} \left(\prod_{j=1}^{m} f_j \left(\prod_{i=1}^{\ell} T_i^{[a_{i,j}(n)]} x \right) - \prod_{j=1}^{m} f_j \left(\prod_{i=1}^{\ell} T_i^{[a_{i,j}(n)+b_{i,j}]} x \right) \right) d\lambda^{\ell m} \Big|.$$

Once more, we study the case where some $\{a_{i,j}(n)\} \geq 1-\delta$. For $1 \leq i \leq \ell$, $1 \leq j \leq m$, let $E_\delta^{i,j} := \{n \in \mathbb{N}\colon \{a_{i,j}(n)\} \in [1-\delta, 1)\}$. Since $\mathbf{1}_{E_\delta^{i,j}}(n) = \mathbf{1}_{[1-\delta,1)}(\{a_{i,j}(n)\})$, we have

$$\frac{1}{\delta^{\ell m}} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} (\delta^{\ell m} a(n) - a'(n)) \right\|_{L^2(\mu)} \leq 2 \sum_{(i,j)\in[1,\ell]\times[1,m]} \frac{|E_\delta^{i,j} \cap [M,N)|}{N-M}.$$

The right-hand side term of this relation is as small as we want, for sufficiently small δ, by condition (ii). Since,

$$\delta^{\ell m} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} a(n) \right\|_{L^2(\mu)} \leq \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} (\delta^{\ell m} a(n) - a'(n)) \right\|_{L^2(\mu)} + \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \tilde{a}(n) \right\|_{L^2(\nu)},$$

we have that

$$\left\| \frac{1}{N-M} \sum_{n=M}^{N-1} a(n) \right\|_{L^2(\mu)} \leq c_\delta + \delta^{-\ell m} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \tilde{a}(n) \right\|_{L^2(\nu)},$$

where $c_\delta \to 0$ as $\delta \to 0^+$. We first take $\limsup_{N-M\to\infty}$ and then $\delta \to 0^+$ to get the result. ☐

In the elementary estimate that follows, we use the notation $\text{U-}\limsup_{n_1,\ldots,n_k} \mathbb{E}_{n_1,\ldots,n_k}$ instead of $\limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n_1=M}^{N-1} \ldots \limsup_{N-M\to\infty} \frac{1}{N-M} \sum_{n_k=M}^{N-1}$.

Lemma 5.2. *Let $k \in \mathbb{N}$ and $s \in (0, +\infty)$. For any sequence $(a(n_1, \ldots, n_k))$ of real non-negative numbers we have*

$$\text{U-}\limsup_{n_1,\ldots,n_k} \mathbb{E}_{n_1,\ldots,n_k} a([n_1 s], \ldots, [n_k s]) \leq s^k \left(\left[\frac{1}{s}\right] + 1 \right)^k \text{U-}\limsup_{n_1,\ldots,n_k} \mathbb{E}_{n_1,\ldots,n_k} a(n_1, \ldots, n_k).$$

Proof. For $k = 1$ we have

$$\frac{1}{N-M} \sum_{n=M}^{N-1} a([ns]) \leq \left(\left[\frac{1}{s}\right] + 1 \right) \frac{[(N-1)s] - [Ms]}{N-M} \frac{1}{[(N-1)s] - [Ms]} \sum_{n=[Ms]}^{[(N-1)s]} a(n).$$

Since $\lim_{N-M\to\infty} \frac{[(N-1)s] - [Ms]}{N-M} = s$, taking $\limsup_{N-M\to\infty}$ will give us the desired inequality.

The general $k > 1$ case follows analogously with induction. ☐

For $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ a system and $f_1, \ldots, f_m \in L^\infty(\mu)$, recall the action $\prod_{i=1}^{\ell} T_{i,a_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,a_{i,m}}$ and the Y-extensions $\hat{f}_j$ of f_j where $Y = X \times [0,1)^{\ell m}$ was endowed with the probability measure $\nu = \mu \times \lambda^{\ell m}$ defined in the proofs of Theorems 3.2, 4.1 and 5.1. By the definition of the action, the first coordinate of $T_{i_0, a_{i_0,j_0}}$ evaluated at the point $(x, b_{1,1}, \ldots, b_{\ell,1}, b_{1,2}, \ldots, b_{\ell,m}) \in Y$ is $T_{i_0}^{[a_{i_0,j_0} + b_{i_0,j_0}]} x$, the $((j_0 - 1)\ell + i_0 + 1)$-coordinate is equal to $\{a_{i_0,j_0} + b_{i_0,j_0}\}$, while in all the other coordinates we have the identity map mapping $b_{i,j}$ to itself. So, without loss of generality, to study the transformations $T_{i,s}$, $s \in \mathbb{R}$, essential to the implications in Section 6, we restrict our study to the $\ell = m = 1$ case: $S = T_s$ where $s \in (0, +\infty)$ (in the case where $s < 0$, we set $S = T_{-s}^{-1}$). Hence, we study the transformation $S(x, b) = (T^{[s+b]}x, \{s + b\})$, for which we have $S^n(x, b) = (T^{[ns+b]}x, \{ns + b\})$, and so, $S^n \hat{f}(x, b) = T^{[ns+b]}f(x)$, since $[s + \{s'\}] + [s'] = [s + s']$ and $\{s + \{s'\}\} = \{s + s'\}$ for all s.

Lemma 5.3. *For any $k \in \mathbb{N}$, there exists a constant $c = c(k, s)$ such that*

$$\|\hat{f}\|_{k,\nu,S} \leq c \|f\|_{k+1,\mu,T}.$$

Proof. Let $c_k = (k+1)^{2^k}$, $c_{k,s} = c_k \cdot s^k \left(\left[\frac{1}{s} \right] + 1 \right)^k$, $F = f \otimes \bar{f}$ and $R = T \times T$. By Lemma 5.2 and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}
\|\hat{f}\|_{k,\nu,S}^{2^k} &= \text{U-}\lim_{n_1,\dots,n_k} \mathbb{E}_{n_1,\dots,n_k} \int \prod_{\vec{\epsilon} \in \{0,1\}^k} \mathcal{C}^{|\epsilon|} S^{\vec{\epsilon}\cdot\vec{n}} \hat{f}\, d\nu \\
&\leq \text{U-}\limsup_{n_1,\dots,n_k} \mathbb{E}_{n_1,\dots,n_k} \left| \int \prod_{\vec{\epsilon} \in \{0,1\}^k} \mathcal{C}^{|\epsilon|} T^{[(\vec{\epsilon}\cdot\vec{n})s+b]} f\, d\nu \right| \\
&\leq c_k \max_{e_{\vec{\epsilon}} \in \{0,\dots,k\}} \text{U-}\limsup_{n_1,\dots,n_k} \mathbb{E}_{n_1,\dots,n_k} \left| \int \prod_{\vec{\epsilon} \in \{0,1\}^k} \mathcal{C}^{|\epsilon|} T^{\sum_{i=1}^k [\epsilon_i n_i s] + e_{\vec{\epsilon}}} f\, d\mu \right| \\
&\leq c_{k,s} \max_{e_{\vec{\epsilon}} \in \{0,\dots,k\}} \text{U-}\limsup_{n_1,\dots,n_k} \mathbb{E}_{n_1,\dots,n_k} \left| \int \prod_{\vec{\epsilon} \in \{0,1\}^k} \mathcal{C}^{|\epsilon|} T^{\vec{\epsilon}\cdot\vec{n} + e_{\vec{\epsilon}}} f\, d\mu \right| \\
&\leq c_{k,s} \max_{e_{\vec{\epsilon}} \in \{0,\dots,k\}} \left(\text{U-}\limsup_{n_1,\dots,n_k} \mathbb{E}_{n_1,\dots,n_k} \left| \int \prod_{\vec{\epsilon} \in \{0,1\}^k} \mathcal{C}^{|\epsilon|} T^{\vec{\epsilon}\cdot\vec{n} + e_{\vec{\epsilon}}} f\, d\mu \right|^2 \right)^{1/2} \\
&= c_{k,s} \max_{e_{\vec{\epsilon}} \in \{0,\dots,k\}} \left(\text{U-}\limsup_{n_1,\dots,n_k} \mathbb{E}_{n_1,\dots,n_k} \int \prod_{\vec{\epsilon} \in \{0,1\}^k} \mathcal{C}^{|\epsilon|} R^{\vec{\epsilon}\cdot\vec{n} + e_{\vec{\epsilon}}} F\, d(\mu \times \mu) \right)^{1/2},
\end{aligned}$$

using relation (10) from [9] and condition (1) of Lemma 3.9 from [9], this last term is bounded by

$$c_{k,s} \max_{e_{\vec{\epsilon}} \in \{0,\dots,k\}} \left(\prod_{\vec{\epsilon} \in \{0,1\}^k} \|R^{e_{\vec{\epsilon}}} F\|_{k,\mu\times\mu,R} \right)^{1/2} = c_{k,s} \left(\|F\|_{k,\mu\times\mu,R}^{2^k} \right)^{1/2} \leq c_{k,s} \|f\|_{k+1,\mu,T}^{2^k},$$

where we have used the fact that $\|F\|_{k,\mu\times\mu,R} = \|f \otimes \bar{f}\|_{k,\mu\times\mu,T\times T} \leq \|f\|_{k+1,\mu,T}^2$. □

Let $(X, \mathcal{X}, \mu)$ be a probability space. If $(T_t)_{t\in\mathbb{R}}$ is a measure preserving flow and $p(t) = b_r t^r + \cdots + b_1 t + b_0 \in \mathbb{R}[t]$, we write

$$T_{p(t)} = T_{b_r}^{t^r} \cdot \ldots \cdot T_{b_1}^{t} T_{b_0}. \tag{15}$$

We assign to the polynomial $p(t)$ an $(r+1)$-tuple of polynomials $\vec{p} = (p^r(t), \dots, p^0(t))$, where we put $p^i(t) = t^i$ if $b_i \neq 0$ and $p^i(t) = 0$ otherwise.

Definition. Let $\ell, m, d \in \mathbb{N}$. The family of m polynomial ℓ-tuples of polynomials with maximum degree d in $\mathbb{R}[t]$, $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)_m = ((p_{1,1}, \ldots, p_{\ell,1}), \ldots, (p_{1,m}, \ldots, p_{\ell,m}))$ is called a *d-nice family in* $\mathbb{R}[t]$ if the respective family of m polynomial $(d+1)\ell$-tuples

$$(\vec{\mathcal{P}}_1, \ldots, \vec{\mathcal{P}}_\ell)^d_m := ((\vec{p}_{1,1}, \ldots, \vec{p}_{\ell,1}), \ldots, (\vec{p}_{1,m}, \ldots, \vec{p}_{\ell,m})),$$

where we complete the coordinates with zeros from the right so that every vector $\vec{p}_{i,j}$ has $d+1$ coordinates, is a nice family in $\mathbb{Z}[t]$ (recall the definition from Subsection 2.2).

Proposition 5.4. *Let $\ell, m, d \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)_m$ be a d-nice family of ℓ-tuples of polynomials in $\mathbb{R}[t]$ with maximum degree d and $f_1, \ldots, f_m \in L^\infty(\mu)$. There exists $k = k(d, \ell, m) \in \mathbb{N}$ such that if $|||f_1|||_{k,\mu,T_1} = 0$, then*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} \left(\prod_{i=1}^{\ell} T_i^{[p_{i,1}(n)]} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[p_{i,m}(n)]} \right) f_m \right\|_{L^2(\mu)} = 0.$$

Proof. Let the action $\prod_{i=1}^{\ell} T_{i,a_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,a_{i,m}}$, where $a_{i,j} = p_{i,j}$, and the Y-extensions $\hat{f}_j$ of f_j be as in the proof of Theorem 3.2. It suffices to show only that condition (i) of Theorem 5.1 holds since condition (ii) holds automatically for real polynomials. According to the hypothesis, there exists $k = k(d, \ell, m) \in \mathbb{N}$ such that $|||f_1|||_{k,\mu,T_1} = 0$. Using Lemma 5.3, we have that $|||\hat{f}_1|||_{k-1,\nu,S_1} = 0$, where, if $b_r^{1,1}$ is the leading coefficient of $p_{1,1}$, S_1 can be chosen to be equal to $T_{1,|b_r^{1,1}|}$. Writing each $T_{i,p_{i,j}}$ as in (15) and using the fact that $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)_m$ is a d-nice family in $\mathbb{R}[t]$, we get the respective nice family $(\vec{\mathcal{P}}_1, \ldots, \vec{\mathcal{P}}_\ell)^d_m$ in $\mathbb{Z}[t]$. The result now follows from Proposition 2.3. ☐

6. Proof of main results and implications

In this last section we prove Theorems 1.1, 1.2 and give some applications. To prove the special linear case of Theorem 1.1, we will make use of the following Hilbert-space variant of van der Corput's estimate:

Lemma 6.1 ([7]). *Let (v_n) be a bounded sequence of vectors in an inner product space and (I_N) be a sequence of intervals in $\mathbb{N}$ with lengths tending to infinity. Then*

$$\limsup_{N\to\infty} \left\| \frac{1}{|I_N|} \sum_{n\in I_N} v_n \right\|^2 \leq 4 \limsup_{H\to\infty} \frac{1}{H} \sum_{h=1}^{H} \limsup_{N\to\infty} \left| \frac{1}{|I_N|} \sum_{n\in I_N} \langle v_{n+h}, v_n \rangle \right|.$$

Proof of Theorem 1.1. General case: Since any sequence $(a(n))$ of the form (1) is k-weak-anti-uniform and k-regular for some $k = k(\ell, m, \max\deg(p_{i,j}))$ as we showed at the end of Sections 3 and 4, we can apply Theorem 2.1 to get the conclusion.

Linear case: According to Theorem 2.1, we have to show $(m+1)$-regularity $(m+1)$-weak-anti-uniformity of the sequence $\int f_0 \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,1}n]} \right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} T_i^{[a_{i,m}n]} \right) f_m \, d\mu.$

The $(m+1)$-regularity follows from Section 3, using Proposition 3.1 and Theorem 1.3.

Real valued polynomials satisfy condition (ii) of Theorem 4.1, so if $S_{i,j} = T_{i,a_{i,j}}$ for every i, j, it suffices to prove that the sequence $\tilde{a}(n) := \int f_0 \cdot \left(\prod_{i=1}^{\ell} S_{i,1}^n\right) f_1 \cdot \ldots \cdot \left(\prod_{i=1}^{\ell} S_{i,m}^n\right) f_m \, d\mu$ is $(m+1)$-anti-uniform. This follows from Subsection 2.3.1 of [7] with induction by successively applying Lemma 6.1 and using the van der Corput operation (at most) m times to reduce the linear polynomial iterates that appear in $\tilde{a}(n)$ to constant ones. $\square$

Proof of Theorem 1.2. From the definitions, we have $\overline{\mathcal{B}_k} \subseteq \overline{\mathcal{C}_k}$, and the linear case of Theorem 1.1 gives us that $\overline{\mathcal{C}_k} \subseteq \overline{\mathcal{A}_k}$. The result follows since $\overline{\mathcal{A}_k} = \overline{\mathcal{B}_k}$ ([7, Theorem 1.4]). $\square$

Theorem 6.2. *For $\ell \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $b_1, \ldots, b_\ell \in \mathbb{R} \setminus \{0\}$, $r_1, \ldots, r_\ell \in \mathbb{N}$ be distinct positive integers and $f_1, \ldots, f_\ell \in L^\infty(\mu)$. There exists $k = k(\ell, \max r_i) \in \mathbb{N}$ such that if $|||f_i|||_{k,\mu,T_i} = 0$ for some $1 \leq i \leq \ell$, then*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} T_1^{[b_1 n^{r_1}]} f_1 \cdot \ldots \cdot T_\ell^{[b_\ell n^{r_\ell}]} f_\ell \right\|_{L^2(\mu)} = 0.$$

Proof. Analogously to Proposition 5.4, for the action $T_{1,b_1 n^{r_1}} \cdot \ldots \cdot T_{\ell,b_\ell n^{r_\ell}}$ (we have set $a_{i,i} = b_i n^{r_i}$ and $a_{i,j} = 0$ for all $i \neq j$) and the Y-extensions of $\hat{f}_j$ of f_j, $1 \leq j \leq \ell$, all defined in the proof of Theorem 3.2, we have only to show that condition (i) of Theorem 5.1 holds. According to the hypothesis, there exists $k = k(d, \max r_i) \in \mathbb{N}$ such that $|||f_{i_0}|||_{k,\mu,T_{i_0}} = 0$ for some $1 \leq i_0 \leq \ell$. Using Lemma 5.3, we have that $|||\hat{f}_{i_0}|||_{k-1,\nu,S_{i_0}} = 0$ where S_{i_0} can be chosen to be equal to $T_{i_0,|b_{i_0}|}$. Since $T_{i,b_i n^{r_i}} = T_{i,b_i}^{n^{r_i}}$ and all r_i are distinct, condition (i) of Theorem 5.1 follows from Theorem 2.2. $\square$

Assuming slightly more than in Theorem 6.2, we have the following result that applies to a larger class of polynomial iterates:

Theorem 6.3. *For $\ell \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be non-constant polynomials with distinct degrees and highest degree $d = \deg(p_1)$ and $f_1, \ldots, f_\ell \in L^\infty(\mu)$. There exists $k = k(\ell, d) \in \mathbb{N}$ such that if $|||f_1|||_{k,\mu,T_1} = 0$, then*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} T_1^{[p_1(n)]} f_1 \cdot \ldots \cdot T_\ell^{[p_\ell(n)]} f_\ell \right\|_{L^2(\mu)} = 0.$$

Proof. Let $(\mathcal{P}_1, \ldots, \mathcal{P}_\ell)_m = ((p_1, 0, \ldots, 0), (0, p_2, \ldots, 0), \ldots, (0, \ldots, 0, p_\ell))$. Since this is a d-nice family in $\mathbb{R}[t]$, we can apply Proposition 5.4. $\square$

Definition ([8])**.** Let $(X, \mathcal{X}, \mu, T)$ be a system. T is called *weakly mixing transformation* and $(X, \mathcal{X}, \mu, T)$ a *weakly mixing system* if for any two functions $f, g \in L^2(\mu)$ we have that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \left| \int f T^n g \, d\mu - \int f \, d\mu \int g \, d\mu \right| = 0.$$

It is an immediate consequence of the definition of the seminorms $|||\cdot|||_k$ that for weakly mixing systems $(X, \mathcal{X}, \mu, T)$, we have $|||f|||_{k,\mu,T} = \left| \int f \, d\mu \right|$ for every $k \in \mathbb{N}$.

Corollary 6.4. *For $\ell \in \mathbb{N}$, let $(X, \mathcal{X}, \mu, T_i)$, $1 \leq i \leq \ell$, be commuting weakly mixing systems, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be non-constant polynomials with distinct degrees and $f_1, \ldots, f_\ell \in L^\infty(\mu)$. Then*

$$\lim_{N-M\to\infty} \left\| \frac{1}{N-M} \sum_{n=M}^{N-1} T_1^{[p_1(n)]} f_1 \cdot \ldots \cdot T_\ell^{[p_\ell(n)]} f_\ell - \prod_{i=1}^{\ell} \int f_i \, d\mu \right\|_{L^2(\mu)} = 0.$$

Proof. We can assume that some $\int f_i \, d\mu = 0$ by using the elementary identity from [8]: $\prod_{i=1}^{\ell} a_i - \prod_{i=1}^{\ell} b_i = \sum_{j=1}^{k} \left(\prod_{i=1}^{j-1} a_i \right) (a_j - b_j) \left(\prod_{i=j+1}^{k} b_i \right)$, where $\prod_{i=1}^{0} a_i = \prod_{i=k+1}^{k} b_i := 1$. Without loss of generality, we may assume that $\int f_{i_0} \, d\mu = 0$, where $\deg(p_{i_0}) = \max \deg(p_i)$. This means $|||f_{i_0}|||_{k,\mu,T_{i_0}} = 0$ for all k. The result now follows from Theorem 6.3. □

Remark. It is an open problem to determine if the conclusion of Corollary 6.4 still holds in the case that the non-constant polynomials p_i are essentially distinct (i.e., $p_i - p_j$ is non-constant for $i \neq j$).

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