INTEGER PART INDEPENDENT POLYNOMIAL AVERAGES AND APPLICATIONS ALONG PRIMES

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Abstract. Exploiting the equidistribution properties of polynomial sequences, following the methods developed by Leibman ([23]) and Frantzikinakis ([8] and [9]), we show that the ergodic averages with iterates given by the integer part of strongly independent real valued polynomials converge in mean to the expected limit. These results have, via Furstenberg's correspondence principle, immediate combinatorial applications, while combining these results with methods from [14] and [21] we get the respective expected limits and combinatorial results for multiple averages for a single sequence, as well as for several sequences along prime numbers.

1. Introduction

The study of the limiting behaviour in $L^2(\mu)$ of multiple ergodic averages of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)}f_1 \cdot \ldots \cdot T^{a_\ell(n)}f_\ell, \tag{1}$$

where $(a_1(n))_n, \ldots, (a_\ell(n))_n$ are sequences of integers, $T : X \to X$ is an invertible measure preserving transformation on the probability space $(X, \mathcal{B}, \mu)$[1] and $f_1, \ldots, f_\ell \in L^\infty(\mu)$ has been of great importance in the area of ergodic theory. The originator of this was Furstenberg (in [16]) who first studied averages as in (1) in the case where $a_i(n) = in$, $1 \leq i \leq \ell$, in order to provide an ergodic theoretical proof of Szemerédi's theorem on arithmetic progressions.

Bergelson and Liebman in [6] studied the case where a_i are integer polynomials with no constant term proving a polynomial extension of Szemerédi's theorem.

Another question that arises studying the limiting behaviour of (1) is if the limit exists what can we say about it. In this direction and for the case of weakly mixing systems Fustenberg, Katznelson and Ornstein proved in [17] that if $a_i(n) = in$, $1 \leq i \leq \ell$, the multiple ergodic averages in (1) converge to the product of integrals of f_i's. Bergelson in [2] extended this result in the case where a_i's are essentially distinct integer polynomials (i.e., all polynomials and their differences are not constant). Furthermore, Frantzikinakis and Kra (in [15]) established the same result under the total ergodicity assumption of the system for a_i's independent integer polynomials (i.e., every non-trivial integral combination of a_i's is not constant). Precise knowledge of the limit for a general system would a priori imply "nice" recurrence and combinatorial results.

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[1]We shall call the quadruple $(X, \mathcal{B}, \mu, T)$ *system.*

The convergence of multiple ergodic averages with several commuting transformations has been studied as well. Under weaker assumptions than the weakly mixing one on the system, the convergence to the expected limit in the case of iterates of integer polynomials (i.e., polynomials with rational coefficients that take integer values at integers) with different degrees and in the case of the integer parts of real polynomials with different degrees is treated in [7] and [20] respectively, where by *expected limit* we mean that under the ergodicity assumption, the limit is equal to the product of integrals of f_i's.

There are also results in other, non-polynomial classes of iterates. Bergelson and Håland-Knutson in [3] treated the case of iterates of integer part tempered functions on a weakly mixing system, while Frantzikinakis treated the case of iterates of integer part logarithmico-exponential Hardy field functions of different growth with polynomial degrees (in [8] for a single T and [11] for multiple commuting T_i's where he also showed that in the sublinear, 0-degree case, the commutativity assumption of T_i's can be removed). All of these results show convergence to the expected limit. We highlight at this point that the last results of Frantzikinakis are very strong and hold for general systems, a behavior that we didn't have for any class of polynomial iterates, with multiple polynomials of degree greater than 1, so far (for a single polynomial see the Theorem 2.10 of Frantzikinakis below).

In this paper we study the convergence of multiple ergodic averages with integer part of real polynomial iterates of several sequences of the form

$$\frac{1}{N}\sum_{n=1}^{N} T^{[p_1(n)]}f_1\cdot\ldots\cdot T^{[p_\ell(n)]}f_\ell. \tag{2}$$

We show in our main theorem, Theorem 2.1, that for the polynomial sequences $(p_1(n))_n$, $\ldots,(p_\ell(n))_n$, where every non-trivial linear combination of p_i's has at least one non-constant irrational coefficient, under the assumption of ergodicity on T (i.e., there is no non-trivial set invariant under T), the limit of (2) as $N\to\infty$ in $L^2(\mu)$ is the expected one, meaning equal to $\prod_{i=1}^{\ell}\int f_i\ d\mu$. So, using the ergodic decomposition, we have for a general system that the aforementioned limit is equal to the product of the conditional expectations of f_i's, i.e., equal to $\prod_{i=1}^{\ell}\mathbb{E}(f_i|\mathcal{I}(T))$. This is arguably the first time that we get existence and precise expression of the limit for general multiple polynomial expressions. Adding the fact that we have no assumptions on our system, even though it is presented via ergodic theory language, our result is a combinatorial object. Measure theoretical and hence, via Furstenberg's correspondence principle, combinatorial applications of Theorem 2.1 are given in Theorems 2.2, 2.3 and 2.4. The strong nature of our results is also reflected in Theorem 2.5, Corollary 2.6, Theorem2.8 and Corollary 2.9, where we obtain some additional applications in topological dynamics and combinatorics respectively.

In order to prove Theorem 2.1 we follow Frantzikinakis' approach (from [8] and [9]) and we also use some results of Leibman ([23]) and Host-Kra ([19]). More specifically, via Lemma 5.1 (Lemma 4.7 in [8]), which informs us that the nilfactor of our system is also the characteristic factor for the family of polynomials of interest, and the Structure theorem, Theorem 3.2, of Host and Kra (which follows by Theorem 10.1 in [19]), it suffices to show Theorem 2.1 when our system is a nilsystem. To complete the proof, we use Theorem 4.1, an equidistribution result first proved by Frantzikinakis in [9] for logarithmico-exponential Hardy field functions with polynomial degree of different growth. In order to derive Theorem 4.1, we use Theorem 3.1, a central equidistribution result due to Leibman (Theorems B and C from [23]) for a polynomial sequence in a connected and simply connected Lie group.

Combining Theorem 2.10, a result from [8] on multiple convergence of a single real polynomial sequence, and Theorem 2.1 with some results from [21], we get the analogous results in Theorems 2.12 and 2.14 respectively, together with their implications, for averages along prime numbers.

Lastly, we show, in Theorems 2.18 and 2.19, the corresponding recurrence results along shifted primes ($\mathbb{P}-1$ and $\mathbb{P}+1$) together with their combinatorial implications via Furstenberg's correspondence principle, in Theorems 2.20 and 2.21.

Throughout the article we will highlight at some points the fact that one cannot expect to obtain the nice convergence and recurrence results that we get for other classical families of polynomials, say, integer polynomials, not even for very special families of them. This fact "forces" one to deal with real-valued polynomial families which satisfy some "Weyl-type" assumptions (see next section).

Notation. With $\mathbb{P}$, $\mathbb{N}=\{1,2,\ldots\}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ we denote the set of prime, natural, integer, rational, real and complex numbers respectively. For $N\in\mathbb{N}$ we write $[1,N]$ to denote the set $\{1,\ldots,N\}$. For a measurable function f on a measure space X with a transformation $T:X\to X$, we denote with Tf the composition $f\circ T$. $\mathbb{T}^s=\mathbb{R}^s/\mathbb{Z}^s$ denotes the s dimensional torus, $e(t)=e^{2\pi i t}$ denotes the exponential map, $(a(n))_n$ denotes a sequence indexed over the natural numbers (i.e., $(a(n))_{n\in\mathbb{N}}$), and $[\cdot]$ denotes the integer part (floor) function.

2. Main results

It follows as a special case of Theorem 1.3 in [20] that for any $\ell\in\mathbb{N}$, $(X,\mathcal{B},\mu,T)$ system, $p_1,\ldots,p_\ell$ real polynomials and functions $f_1,\ldots,f_\ell\in L^\infty(\mu)$ the limit

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}T^{[p_1(n)]}f_1\cdot\ldots\cdot T^{[p_\ell(n)]}f_\ell \tag{3}$$

exists in $L^2(\mu)$.

We now define a notion of independence of real polynomials that will ensure convergence of (3) to the expected limit (see Theorem 2.1 below).

Definition. For $\ell\in\mathbb{N}$, let $\{p_1,\ldots,p_\ell\}$ be a family of real polynomials. We say that this family is *strongly independent* (or that the polynomials $p_1,\ldots,p_\ell$ are *strongly independent*) if any non-trivial real linear combination of the polynomials p_i has a non-constant irrational coefficient.

Note that a family with one element, $\{p\}$, where $p\in\mathbb{R}[t]$, is strongly independent iff $p(t)\neq cq(t)+d$ for all $c,d\in\mathbb{R}$ and $q\in\mathbb{Q}[t]$ (or $\mathbb{Z}[t]$ equivalently).

Example. The family of polynomials $\{\sqrt{2}t^3+t^2, \sqrt{3}t^3-t\}$ is strongly independent while the families $\{\sqrt{5}t^3+t^2+\sqrt{6}t, t^2, \sqrt{7}t\}$ and $\{\sqrt{2}t^2+t, \sqrt{5}t^2-t\}$ are not.

We also remark that our definition coincides with the definition of the *good* family of polynomials given in [10], Problem 10, in the special case where the polynomial sequences are constant.

Via our method of proof, it will later become clear that the assumptions on the real polynomials that we have are in a sense "optimal", since they are exactly what one has to assume in order to obtain the Weyl-type equidistribution results we have to show in order to prove our main result, i.e., that the limit of the ergodic averages over strongly independent real polynomials exists and it is the expected one.

Theorem 2.1. *Let $\ell \in \mathbb{N}$, $p_1,\ldots,p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials, $(X,\mathcal{B},\mu,T)$ be an ergodic system and $f_1,\ldots,f_\ell \in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N} T^{[p_1(n)]}f_1\cdot\ldots\cdot T^{[p_\ell(n)]}f_\ell = \prod_{i=1}^{\ell}\int f_i\, d\mu, \tag{4}$$

where the convergence takes place in $L^2(\mu)$.

Remark. The assumption that the polynomials are strongly independent is necessary, since even for $\ell=1$, $p(t)=\sqrt{2}t$ and ergodic rotations on the torus, (4) typically fails.

In the study of ergodic averages with polynomial iterates, some important families are the ones of finitely many *independent, integer polynomials* (i.e., any non-trivial linear combination of integer coefficients is non-constant). We note that even for such families it is not true in general that one has convergence as in (4), i.e., to the expected limit for a general ergodic system (see remark after Theorem 2.2). Such a result requires more assumptions on the system, as the total ergodicity one (see [15]). Hence, one is "forced" to work with real polynomials in order to have this nice convergence behavior.

We now state a principle due to Furstenberg which allows one to obtain combinatorial results from ergodic theoretical ones. We present here a formulation of this result from [1].

Theorem (Furstenberg correspondence principle, [16], Theorem 1.1, [1])**.** *Let E be a subset of integers. There exists a system $(X,\mathcal{B},\mu,T)$ and a set $A\in\mathcal{B}$, with $\mu(A)=\bar{d}(E)$ [2] such that*

$$\bar{d}(E\cap(E-n_1)\cap\ldots\cap(E-n_\ell)) \geq \mu(A\cap T^{-n_1}A\cap\ldots\cap T^{-n_\ell}A)$$

for every $\ell\in\mathbb{N}$ and $n_1,\ldots,n_\ell\in\mathbb{Z}$.

As a consequence of Theorem 2.1 we get the following recurrence result (for a proof of this implication see for example Theorem 2.8 in [8]).

Theorem 2.2. *Let $\ell\in\mathbb{N}$ and $p_1,\ldots,p_\ell\in\mathbb{R}[t]$ be strongly independent real polynomials. Then for every system $(X,\mathcal{B},\mu,T)$ and $A\in\mathcal{B}$ we have*

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu\Big(A\cap T^{-[p_1(n)]}A\cap\ldots\cap T^{-[p_\ell(n)]}A\Big) \geq (\mu(A))^{\ell+1}. \tag{5}$$

[2] For a set $A\subseteq\mathbb{N}$ we define its *upper density*, $\bar{d}(A)$, as $\bar{d}(A)=\limsup_{N\to\infty}\frac{|A\cap\{1,\ldots,N\}|}{N}$. If the limit of the previous expression exists as $N\to\infty$, we say that its value, denoted with $d(A)$, is the *density* of A.

Remark. The assumption that the polynomials are strongly independent is necessary since even for $\ell = 1$ and $p(t) = t^2$, (5) typically fails.

So, the previous remark shows that (5) typically fails even for families of independent, integer polynomials. Hence, Theorem 2.2 is another indication that one has to work with real polynomials in order to have nice lower bounds as in (5) for general systems.

Note at this point that our arguments show that the uniform version of Theorem 2.1, and hence its implications, holds, meaning that one can replace the standard Cesàro averages, $\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}$, with the respective uniform ones, $\lim_{N-M\to\infty}\frac{1}{N-M}\sum_{n=M+1}^{N}$, and the natural upper density, $\bar{d}$, with the respective upper Banach density, d^* [3].

Then, one has that the uniform version of Theorem 2.2 implies that for any $A \in \mathcal{B}$ with $\mu(A) > 0$, and every $\varepsilon > 0$ the set

$$R_\varepsilon(A) = \left\{n \in \mathbb{Z} :\ \mu\Big(A \cap T^{-[p_1(n)]}A \cap \ldots \cap T^{-[p_\ell(n)]}A\Big) > (\mu(A))^{\ell+1} - \varepsilon\right\}$$

is syndetic (i.e., it has bounded gaps).

We note that this general result, which holds under no assumption on the system, implies that a family of strongly independent real polynomials has a much different behavior than a family of linear integer polynomials, since its stands in contrast with the Bergelson-Host-Kra-Ruzsa counterexample to the "higher-order Khintchine recurrence theorem". Indeed, in [4], the aforementioned authors found an ergodic system $(X, \mathcal{B}, \mu, T)$ and a set $A \in \mathcal{B}$ with $\mu(A) > 0$ such that

$$\mu\Big(A \cap T^{-n}A \cap T^{-2n}A \cap T^{-3n}A \cap T^{-4n}A\Big) \leq \frac{\mu(A)^5}{2} \quad \text{for all} \ \ n \neq 0$$

(so, for $p_i(t) = it$ we have that the syndeticity conclusion of the respective $R_\varepsilon(A)$ fails for certain ergodic systems when $\ell \geq 4$, while for certain non-ergodic systems it fails even when $\ell \geq 2$. For examples covering both cases, see [4]).

Using Theorem 2.2 and Furstenberg's corresponding principle, we have the following.

Theorem 2.3. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Then for every $E \subseteq \mathbb{N}$ we have*

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\bar{d}(E \cap (E - [p_1(n)]) \cap \ldots \cap (E - [p_\ell(n)])) \geq (\bar{d}(E))^{\ell+1}.$$

An immediate implication of the aforementioned result is the following.

Theorem 2.4. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Then every $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$ contains arithmetic configurations of the form*

$$\{m, m + [p_1(n)], m + [p_2(n)], \ldots, m + [p_\ell(n)]\}$$

for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $[p_i(n)] \neq 0$, for all $1 \leq i \leq \ell$.

[3] For a set $A \subseteq \mathbb{Z}$, we define its *upper Banach density*, $d^*(A)$, as $d^*(A) = \limsup_{N-M\to\infty}\frac{|A \cap \{M+1, \ldots, N\}|}{N-M}$.

We note that one can get the aforementioned result for integer polynomials with no constant term from the polynomial Szemerédi theorem (Theorem A_0, [6]), but in the generality that we present it here it is not clear to us at all if the theorem follows from previous results in the literature.

In the next two applications of Theorem 2.1 we follow the Subsections 2.3 and 2.4 from [11] respectively, where we get similar results for sequences of strongly independent polynomials instead of sequences of Hardy field functions.

2.0.1. *An application in topological dynamics.* Let (X,T) be a (topological) dynamical system, where (X,d) is a compact metric space and $T : X \to X$ an invertible continuous transformation. Suppose T is minimal (i.e., $\overline{\{T^n x : \ n \in \mathbb{N}\}} = X$ for all $x \in X$, hence, for every $x \in X$ and non-empty open set U the set $\{n \in \mathbb{N} : \ T^n x \in U\}$ is syndetic). There exists a T-invariant Borel measure which gives positive value to every non-empty open set. So, due to syndeticity, for every $x \in X$ and every non-empty open set U we have

$$\liminf_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_U(T^n x) > 0 \tag{6}$$

(this limit actually exists). Note that from Theorem 2.1, using the ergodic decomposition, it follows that

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{[p_1(n)]} f_1 \cdot \ldots \cdot T^{[p_\ell(n)]} f_\ell = \prod_{i=1}^{\ell} \mathbb{E}(f_i|\mathcal{I}(T)), \tag{7}$$

where the convergence takes place in $L^2(\mu)$, $p_1, \ldots, p_\ell$ are strongly independent real polynomials, $f_1, \ldots, f_\ell \in L^\infty(\mu)$, $\mathcal{I}(T)$ denotes the σ-algebra of T-invariant sets and $\mathbb{E}(f|\mathcal{I}(T))$ is the conditional expectation with respect to $\mathcal{I}(T)$.

Indeed, if $\mu = \int \mu_t \; d\lambda(t)$ denotes the ergodic decomposition of μ, it suffices to show that if $\mathbb{E}(f_i|\mathcal{I}(T)) = 0$ for some i then the averages converge to 0. Since $\mathbb{E}(f_i|\mathcal{I}(T)) = 0$, we have that $\int f_i \; d\mu_t = 0$ for λ-a.e. t. By (4), we have that the averages go to 0 in $L^2(\mu_t)$ for λ-a.e. t, hence the limit is equal to 0 in $L^2(\mu)$.

Since $\mathbb{E}(f_i|\mathcal{I}(T)) = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^n f_i$, combining (7) with (6), we get for almost every $x \in X$ (and hence for a dense set) and every $U_1, \ldots, U_\ell$ from a given countable basis of non-empty open sets that

$$\limsup_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \mathbf{1}_{U_1}(T^{[p_1(n)]} x) \cdot \ldots \cdot \mathbf{1}_{U_\ell}(T^{[p_\ell(n)]} x) > 0. \tag{8}$$

Using this we get:

Theorem 2.5. *Let $\ell \in \mathbb{N}$, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials and (X,T) a minimal dynamical system. Then for a residual and T-invariant set of $x \in X$ we* *have*

$$\overline{\left\{\left(T^{[p_1(n)]}x,\ldots,T^{[p_\ell(n)]}x\right):\ n\in\mathbb{N}\right\}}=X\times\cdots\times X. \tag{9}$$

Proof of Theorem 2.5. Relation (8) immediately implies that the set of points that satisfies (9), say R, is dense. To see that it is G_δ, take $\ell=1$ (the general case is analogous). Then

$$R=\left\{x\in X:\ \text{for all } m,r\in\mathbb{N},\ \text{there exists } n\in\mathbb{N} \text{ with } T^{[p_1(n)]}x\in B(x_m,1/r)\right\},$$

where $\{x_m:\ m\in\mathbb{N}\}$ is a countable, dense subset of X and $B(x_m,1/r)$ denotes the open ball centered at x_m with radius $1/r$. The claim now follows since

$$R=\bigcap_{m,r\in\mathbb{N}}\bigcup_{n\in\mathbb{N}}T^{-[p_1(n)]}B(x_m,1/r).$$

With a standard convergence argument, we can also show that R is T-invariant. □

Remark. Even for $\ell=1$ examples of minimal rotations on finite cyclic groups show that if $p\in\mathbb{Z}[t]$ is any polynomial different than $\pm t+d$, then (9) may fail for every $x\in X$.

Using Zorn's lemma we can easily show that every dynamical system has a minimal subsystem. So, using this and Theorem 2.5 we get:

Corollary 2.6. *Let $\ell\in\mathbb{N}$, $p_1,\ldots,p_\ell\in\mathbb{R}[t]$ be strongly independent real polynomials and (X,T) a dynamical system. Then for a non-empty and T-invariant set of $x\in X$ we have*

$$\overline{\left\{\left(T^{[p_1(n)]}x,\ldots,T^{[p_\ell(n)]}x\right):\ n\in\mathbb{N}\right\}}=\overline{\{T^nx:\ n\in\mathbb{N}\}}\times\cdots\times\overline{\{T^nx:\ n\in\mathbb{N}\}}. \tag{10}$$

Remark. As in the previous remark, examples for $\ell=1$ and $p\in\mathbb{Z}[t]$ with $p(t)\neq\pm t+d$ show that (10) typically fails.

2.0.2. *An application in combinatorics.* Using Theorem 2.1, we have the following recurrence result (for a proof use an argument similar to the one in Theorem 2.4 in [11]):

Theorem 2.7. *Let $\ell\in\mathbb{N}$, $p_1,\ldots,p_\ell\in\mathbb{R}[t]$ be strongly independent real polynomials, $(X,\mathcal{B},\mu,T)$ a system and $A_0,A_1,\ldots,A_\ell\in\mathcal{B}$ such that*

$$\mu(A_0\cap T^{k_1}A_1\cap\ldots\cap T^{k_\ell}A_\ell)=\alpha>0$$

for some $k_1,\ldots,k_\ell\in\mathbb{Z}$. Then

$$\lim_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu\Big(A_0\cap T^{-[p_1(n)]}A_1\cap\ldots\cap T^{-[p_\ell(n)]}A_\ell\Big)\geq\alpha^{\ell+1}.$$

Using this result and a variant of Furstenberg's correspondence principle for several sets A_i (see Proposition 3.3 from [12]) we get (see the $d=1$ case of Theorem 2.8 from [11]):

Theorem 2.8. *Let $\ell\in\mathbb{N}$, $p_1,\ldots,p_\ell\in\mathbb{R}[t]$ be strongly independent real polynomials and $E_0,E_1,\ldots,E_\ell\subseteq\mathbb{N}$ such that*

$$\overline{d}(E_0\cap(E_1+k_1)\cap\ldots\cap(E_\ell+k_\ell))=\alpha>0$$

for some $k_1,\ldots,k_\ell\in\mathbb{Z}$. Then

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\overline{d}(E_0\cap(E_1-[p_1(n)])\cap\ldots\cap(E_\ell-[p_\ell(n)]))\geq\alpha^{\ell+1}.$$

We will sketch the proof of this result. In order to do so we recall a definition from [12]:

Definition (Definition 5, [12])**.** We say that the sequences $a_1, \ldots, a_\ell \in \ell^\infty(\mathbb{Z})$ *admit correlations along the sequence of intervals* $([1, N_k])_k$ with $N_k \to \infty$ as $k \to \infty$, if the limit

$$\lim_{k\to\infty} \frac{1}{N_k} \sum_{n=1}^{N_k} b_1(n+m_1) \cdot \ldots \cdot b_s(n+m_s)$$

exists for every $s \in \mathbb{N}$, $m_1, \ldots, m_s \in \mathbb{Z}$ and all sequences $b_1, \ldots, b_s \in \{a_1, \ldots, a_\ell, \overline{a}_1, \ldots, \overline{a}_\ell\}$.

We remark that for $a_1, \ldots, a_\ell \in \ell^\infty(\mathbb{Z})$, using a diagonal argument, for any sequence $(N_k)_k \subseteq \mathbb{N}$ with $N_k \to \infty$ as $k \to \infty$, we can find a subsequence $(N'_k)_k$ such that $a_1, \ldots, a_\ell$ admit correlations along the sequence of intervals $([1, N'_k])_k$.

Proof of Theorem 2.8. Find a sequence of intervals $\mathbf{N} := ([1, N_k])_k$ along which the upper density of the intersection in the assumption is attained. Let $d_{\mathbf{N}}$ denote the corresponding density. Passing to a subsequence, if needed, which we denote again by $([1, N_k])_k$, we can assume that the functions $\mathbf{1}_{E_0}, \ldots, \mathbf{1}_{E_\ell}$ admit correlations along the sequence $([1, N_k])_k$. Using Proposition 3.3 from [12], we have that there exists a system $(X, \mathcal{B}, \mu, T)$ and sets $A_0, \ldots, A_\ell \in \mathcal{B}$ such that

$$d_{\mathbf{N}}(E_0 \cap (E_1 - m_1) \cap \ldots \cap (E_\ell - m_\ell)) = \mu(A_0 \cap T^{m_1} A_1 \cap \ldots \cap T^{m_\ell} A_\ell)$$

for all $m_1, \ldots, m_\ell \in \mathbb{Z}$. The result now follows by Theorem 2.7. □

This result can be applied to several syndetic sets $E_0, E_1, \ldots, E_\ell \subseteq \mathbb{N}$ with constant $\alpha = \Big(\prod_{i=0}^{\ell} r_i\Big)^{-1}$, where r_i is the syndeticity constant of E_i, $0 \leq i \leq \ell$. So, one immediately gets the following:

Corollary 2.9. *Let $\ell \in \mathbb{N}$, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials and $E_0, E_1, \ldots, E_\ell \subseteq \mathbb{N}$ be syndetic sets. Then there exists $m, n \in \mathbb{N}$ such that*

$$m \in E_0,\ m + [p_1(n)] \in E_1,\ \ldots,\ m + [p_\ell(n)] \in E_\ell.$$

Via this last result, for a syndetic set $E \subseteq \mathbb{N}$, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ real strongly independent polynomials and $c_0, c_1, \ldots, c_\ell \in \mathbb{N}$, setting $E_i = c_i E$,[4] $0 \leq i \leq \ell$, we can find $x_0, x_1, \ldots, x_\ell \in E$ and $n \in \mathbb{N}$, solution of the following system of equations:

$$\begin{aligned} c_1 x_1 - c_0 x_0 &= [p_1(n)] \\ c_2 x_2 - c_0 x_0 &= [p_2(n)] \\ &\vdots \\ c_\ell x_\ell - c_0 x_0 &= [p_\ell(n)]. \end{aligned}$$

Let us note at this point that similar results fail even for $\ell = 1$, i.e., a single polynomial sequence and also fail when the set E is only assumed to be piecewise syndetic. Easy examples show that if $p \in \mathbb{Z}[t]$ is any polynomial different than $\pm t + d$ and $k \in \mathbb{N} \setminus \{1\}$, then the equation $kx - y = p(n)$ has no solution with x, y belonging to some set E that is an arithmetic progression.

[4]Where $cE := \{cn :\ n \in E\}$.

2.1. **Convergence along primes.** Using Theorems 2.10 (see below), 2.1 and some results from [13], [14] and [21], we can prove integer part polynomial multiple convergence along primes to the expected limit for strongly independent polynomial families.

2.1.1. *Single sequence.* The next result informs us that the limit of the ergodic averages with integer part of real polynomial iterates of a single sequence is equal to the limit of the "Furstenberg averages", and it follows by Theorem 2.2 in [8].

Theorem 2.10 ([8]). *Let $p \in \mathbb{R}[t]$ with $p(t) \neq cq(t)+d$, for all $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$. Then for every $\ell \in \mathbb{N}$, system $(X, \mathcal{B}, \mu, T)$ and $f_1, \ldots, f_\ell \in L^\infty(\mu)$, we have*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{[p(n)]} f_1 \cdot T^{2[p(n)]} f_2 \cdot \ldots \cdot T^{\ell[p(n)]} f_\ell = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{n} f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{\ell n} f_\ell, \tag{11}$$

where the convergence takes place in $L^2(\mu)$.

This theorem, via Furstenberg's correspondence principle and Furstenberg's ergodic Szemerédi theorem, immediately implies the following Szemerédi type result:

Theorem 2.11 ([8]). *Let $p \in \mathbb{R}[t]$ with $p(t) \neq cq(t)+d$, where $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$. Then for every $\ell \in \mathbb{N}$, every $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$ contains arithmetic progressions of the form*

$$\{m, m+[p(n)], m+2[p(n)], \ldots, m+\ell[p(n)]\}$$

for some $m \in \mathbb{Z}$ and $n \in \mathbb{N}$ with $[p(n)] \neq 0$.

We will show the respective versions of these two last results along primes.

Theorem 2.12. *Let $q \in \mathbb{R}[t]$ with $q(t) \neq c\tilde{q}(t)+d$ for all $c, d \in \mathbb{R}$ and $\tilde{q} \in \mathbb{Q}[t]$. Then for every $\ell \in \mathbb{N}$, system $(X, \mathcal{B}, \mu, T)$ and $f_1, \ldots, f_\ell \in L^\infty(\mu)$, we have that*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}\cap[1,N]} T^{[q(p)]} f_1 \cdot T^{2[q(p)]} f_2 \cdot \ldots \cdot T^{\ell[q(p)]} f_\ell = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} T^{n} f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{\ell n} f_\ell,$$

where the convergence takes place in $L^2(\mu)$ and $\pi(N) = |\mathbb{P} \cap [1, N]|$ denotes the number of primes up to N.

Theorem 2.13. *Let $q \in \mathbb{R}[t]$ with $q(t) \neq c\tilde{q}(t)+d$, where $c, d \in \mathbb{R}$ and $\tilde{q} \in \mathbb{Q}[t]$. Then for every $\ell \in \mathbb{N}$, every $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$ contains arithmetic progressions of the form*

$$\{m, m+[q(p)], m+2[q(p)], \ldots, m+\ell[q(p)]\}$$

for some $m \in \mathbb{Z}$ and $p \in \mathbb{P}$ with $[q(p)] \neq 0$.

2.1.2. *Several sequences.* We also get the respective result and its applications of Theorem 2.1 along primes:

Theorem 2.14. *Let $\ell \in \mathbb{N}$, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials, $(X, \mathcal{B}, \mu, T)$ be an ergodic system and $f_1, \ldots, f_\ell \in L^\infty(\mu)$. Then*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p \in \mathbb{P}\cap[1,N]} T^{[p_1(p)]} f_1 \cdot \ldots \cdot T^{[p_\ell(p)]} f_\ell = \prod_{i=1}^{\ell} \int f_i \, d\mu,$$

where the convergence takes place in $L^2(\mu)$.

Theorem 2.14 has the following implications, analogous to the ones that Theorem 2.1 has.

Theorem 2.15. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Then for every system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ we have*

$$\lim_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} \mu\Big(A \cap T^{-[p_1(p)]}A \cap \ldots \cap T^{-[p_\ell(p)]}A\Big) \geq (\mu(A))^{\ell+1}.$$

Theorem 2.16. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Then for every $E \subseteq \mathbb{N}$ we have*

$$\liminf_{N\to\infty} \frac{1}{\pi(N)} \sum_{p\in\mathbb{P}\cap[1,N]} \bar{d}(E \cap (E - [p_1(p)]) \cap \ldots \cap (E - [p_\ell(p)])) \geq (\bar{d}(E))^{\ell+1}.$$

Theorem 2.17. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Then every $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$ contains arithmetic configurations of the form*

$$\{m, m + [p_1(p)], m + [p_2(p)], \ldots, m + [p_\ell(p)]\}$$

for some $m \in \mathbb{Z}$ and $p \in \mathbb{P}$ with $[p_i(p)] \neq 0$, for all $1 \leq i \leq \ell$.

2.2. **Recurrence along shifted primes.** In this last subsection we get some recurrence results along shifted primes for the polynomial families of interest. Namely, we show the following:

Theorem 2.18. *Let $\ell \in \mathbb{N}$ and $p \in \mathbb{R}[t]$ with $p(t) \neq cq(t) + d$, where $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$. Then, for every system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, the set of integers n such that*

$$\mu\left(A \cap T^{-[p(n)]}A \cap T^{-2[p(n)]}A \cap \ldots \cap T^{-\ell[p(n)]}A\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

Theorem 2.19. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Then, for every system $(X, \mathcal{B}, \mu, T)$ and $A \in \mathcal{B}$ with $\mu(A) > 0$, the set of integers n such that*

$$\mu\left(A \cap T^{-[p_1(n)]}A \cap \ldots \cap T^{-[p_\ell(n)]}A\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

Via Furstenberg's correspondence principle the previous results imply:

Theorem 2.20. *Let $\ell \in \mathbb{N}$, $p \in \mathbb{R}[t]$ with $p(t) \neq cq(t) + d$, where $c, d \in \mathbb{R}$ and $q \in \mathbb{Q}[t]$ and let $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$. Then the set of integers n such that*

$$\bar{d}(E \cap (E - [p(n)]) \cap (E - 2[p(n)]) \cap \ldots \cap (E - \ell[p(n)])) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

Theorem 2.21. *Let $\ell \in \mathbb{N}$, $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials and let $E \subseteq \mathbb{N}$ with $\bar{d}(E) > 0$. Then the set of integers n such that*

$$\bar{d}(E \cap (E - [p_1(n)]) \cap \ldots \cap (E - [p_\ell(n)])) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

Remark. The arguments will show that the intersections of Theorems 2.18 and 2.19 have positive measure for a set of positive relative density in the shifted primes (and so, the same holds for the conclusions of Theorems 2.20 and 2.21 as well).

We close this section with the remark that we believe all the respective reformulations of the results stated in this section for a single transformation hold for several commuting transformations but the method we use does not allow us to prove any of them in this more general setting.

3. Background Material

3.1. **Nilmanifolds.** In this subsection we recall some basic facts concerning nilmanifolds and equidistribution results on them.

3.1.1. *Definitions and basic properties.* Let G be a k-step nilpotent Lie group, meaning $G_{k+1} = \{e\}$ for some $k \in \mathbb{N}$, where $G_k = [G, G_{k-1}]$ denotes the k-th commutator subgroup, and let Γ be a discrete cocompact subgroup of G. The compact homogeneous space $X = G/\Gamma$ is called a *k-step nilmanifold* (or just *nilmanifold*).

The group G acts on G/Γ by left translations where the translation by an element $b \in G$ is given by $T_b(g\Gamma) = (bg)\Gamma$. We denote by m_X the normalized *Haar measure* on X, meaning the unique probability measure that is invariant under the action of G by left translations and $\mathcal{G}/\Gamma$ denotes the Borel σ-algebra of G/Γ. If $b \in G$, we call the system $(G/\Gamma, \mathcal{G}/\Gamma, m_X, T_b)$ a *k-step nilsystem* (or just *nilsystem*) and the elements of G *nilrotations*.

3.1.2. *Equidistribution on nilmanifolds.* Let $\exp : \mathfrak{g} \to G$ be the exponential map, where $\mathfrak{g}$ is the Lie algebra of G for a connected and simply connected Lie group G. For $b \in G$ and $s \in \mathbb{R}$ we define the element b^s of G as follows: If $Z \in \mathfrak{g}$ is such that $\exp(Z) = b$, then $b^s = \exp(sZ)$ (this is well defined since exp is a bijection).

If $(a(n))_n$ is a sequence of real numbers and $X = G/\Gamma$ is a nilmanifold with G connected and simply connected, we say that the sequence $(b^{a(n)}x)_n$ is *equidistributed* in a sub-nilmanifold Y of X, if for every $F \in C(Y)$ we have

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} F(b^{a(n)}x) = \int F \, dm_Y.$$

If the sequence $(a(n))_n$ takes only integer values, we are not obliged to assume that G is connected and simply connected.

A nilrotation $b \in G$ is *ergodic*, or *acts ergodically* on X, if the sequence $(b^n\Gamma)_n$ is dense in X. If $b \in G$ is ergodic, then for every $x \in X$ the sequence $(b^n x)_n$ is equidistributed in X (a nontrivial fact which follows by unique ergodicity).

Let $X = G/\Gamma$ be a nilmanifold and $b \in G$. Then the orbit closure $\overline{(b^n\Gamma)_n}$ of b has the structure of a nilmanifold. Furthermore, the sequence $(b^n\Gamma)_n$ is equidistributed in $\overline{(b^n\Gamma)_n}$. If G is connected and simply connected and $b \in G$, then $\overline{(b^s\Gamma)_{s\in\mathbb{R}}}$ is a nilmanifold. Furthermore, the nilflow $(b^s\Gamma)_{s\in\mathbb{R}}$ is equidistributed in $\overline{(b^s\Gamma)_{s\in\mathbb{R}}}$.

If G is a nilpotent group, then a sequence $g : \mathbb{N} \to G$ of the form $g(n) = b_1^{p_1(n)} \cdots b_k^{p_k(n)}$, where $b_i \in G$ and p_i are polynomials taking integer values at the integers for every $1 \leq i \leq k$

is called a *polynomial sequence* in G. A *polynomial sequence on the nilmanifold* $X = G/\Gamma$ is a sequence of the form $(g(n)\Gamma)_n$ where $g : \mathbb{N} \to G$ is a polynomial sequence in G.

The following qualitative equidistribution result was established by Leibman in [23]:

Theorem 3.1 (Theorems B, C, [23]). *Suppose that $X = G/\Gamma$ is a nilmanifold with G connected and simply connected and $(g(n))_n$ is a polynomial sequence in G. Let $Z = G/([G,G]\Gamma)$ and $\pi : X \to Z$ be the natural projection. Then the following statements hold:*

(i) *For every $x \in X$ the sequence $(g(n)x)_n$ is equidistributed in a finite union of subnilmanifolds of X.*
(ii) *For every $x \in X$ the sequence $(g(n)x)_n$ is equidistributed in X if and only if the sequence $(g(n)\pi(x))_n$ is equidistributed in Z.*

If $X = G/\Gamma$ is a nilmanifold with G connected and simply connected, then Z is a connected compact abelian Lie group, hence a torus, meaning $\mathbb{T}^s$ for some $s \in \mathbb{N}$, and as a consequence every nilrotation in Z is isomorphic to a rotation on $\mathbb{T}^s$.

3.2. **Ergodic Theory.** We gather below some basic notions and facts from ergodic theory that we use throughout the paper.

3.2.1. *Factors.* A *homomorphism* from a system $(X, \mathcal{X}, \mu, T)$ onto a system $(Y, \mathcal{Y}, \nu, S)$ is a measurable map $\pi : X \to Y$, such that $\mu \circ \pi^{-1} = \nu$ and $S \circ \pi(x) = \pi \circ T(x)$ for $x \in X$. When we have such a homomorphism we say that the system $(Y, \mathcal{Y}, \nu, S)$ is a *factor* of the system $(X, \mathcal{X}, \mu, T)$. If the factor map $\pi : X \to Y$ can be chosen to be injective, then we say that the systems $(X, \mathcal{X}, \mu, T)$ and $(Y, \mathcal{Y}, \nu, S)$ are *isomorphic.* A factor can also be characterised by $\pi^{-1}(\mathcal{Y})$ which is a T-invariant sub-σ-algebra of $\mathcal{X}$. By a classical abuse of terminology we denote by the same letter the σ-algebras $\mathcal{Y}$ and $\pi^{-1}(\mathcal{Y})$.

3.2.2. *Characteristic Factors.* Let $(X, \mathcal{X}, \mu, T)$ be a system. We say that the σ-algebra $\mathcal{Y}$ of $\mathcal{X}$ is a *characteristic factor* for the family of integer sequences $\{(a_1(n))_n, \dots, (a_\ell(n))_n\}$ if $\mathcal{Y}$ is T-invariant and

$$\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)} f_1 \cdot \ldots \cdot T^{a_\ell(n)} f_\ell - \frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)} \tilde{f}_1 \cdot \ldots \cdot T^{a_\ell(n)} \tilde{f}_\ell \right\|_{L^2(\mu)} = 0,$$

where $\tilde{f}_i = \mathbb{E}(f_i|\mathcal{Y})$, for $f_i \in L^\infty(\mu)$ for all $1 \leq i \leq \ell$ [5].

3.2.3. *Seminorms and Nilfactors.* We follow [19] and [7] for the inductive definition of the seminorms $\lVert\!| \cdot |\!\rVert_k$. More specifically, the definition that we use here follows from [19] (in the ergodic case), [7] (in the general case) and the use of von Neumann's ergodic theorem.

Let $(X, \mathcal{B}, \mu, T)$ be a system and $f \in L^\infty(\mu)$. We define inductively the seminorms $\lVert\!| f |\!\rVert_k$ as follows: For $k = 1$ we set

$$\lVert\!| f |\!\rVert_1 := \lVert \mathbb{E}(f|\mathcal{I}(T)) \rVert_{L^2(\mu)} .$$

[5] Equivalently, if $\mathbb{E}(f_i|\mathcal{Y}) = 0$ for some $1 \leq i \leq \ell$, then $\lim_{N\to\infty} \left\| \frac{1}{N}\sum_{n=1}^{N} T^{a_1(n)} f_1 \cdot \ldots \cdot T^{a_\ell(n)} f_\ell \right\|_{L^2(\mu)} = 0.$

For $k \geq 1$, we let

$$\|\!|f|\!\|_{k+1}^{2^{k+1}} := \lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \|\!|\bar{f}\cdot T^n f|\!\|_k^{2^k}.$$

It was shown in [19] that for every integer $k \geqslant 1$ all these limits exist and $\|\!|\cdot|\!\|_k$ defines a seminorm on $L^\infty(\mu)$.

Using these seminorms we can construct factors $\mathcal{Z}_k = \mathcal{Z}_k(T)$ of X characterized by the property:

$$\text{for } f \in L^\infty(\mu), \quad \mathbb{E}(f|\mathcal{Z}_{k-1}) = 0 \text{ if and only if } \|\!|f|\!\|_k = 0.$$

It was also shown in [19] that for every $k \in \mathbb{N}$ the factor $\mathcal{Z}_k$ has an algebraic structure, in fact we can assume that it is a k-step nilsystem. This is the content of the following Structure theorem, which we recall in the ergodic case and follows by Theorem 10.1 in [19]:

Theorem 3.2 (**Host & Kra**, [19])**.** *Let $(X, \mathcal{B}, \mu, T)$ be an ergodic system and $k \in \mathbb{N}$. Then the factor $\mathcal{Z}_k(T)$ is an inverse limit of k-step nilsystems.* [6]

Because of this result we call $\mathcal{Z}_k$ the *k-step nilfactor* of the system. The smallest factor that is an extension of all finite step nilfactors is denoted by $\mathcal{Z} = \mathcal{Z}(T)$, meaning, $\mathcal{Z} = \bigvee_{k\in\mathbb{N}} \mathcal{Z}_k$, and is called the *nilfactor* of the system.

4. Equidistribution Results

In this section we establish some equidistribution results in order to prove the convergence and recurrence results stated in Section 2. In order to obtain these equidistribution results we follow the main strategy introduced in [9] (Sections 4 and 5).

First, we give an equidistribution result involving nil-orbits of several sequences of strongly independent polynomials (first proved for Hardy field functions [9, Theorem 1.3]).

Theorem 4.1. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials.*

(i) *If $X_i = G_i/\Gamma_i$, $1 \leq i \leq \ell$, are nilmanifolds with G_i connected and simply connected, then for every $b_i \in G_i$ and $x_i \in X_i$ the sequence*

$$\left(b_1^{p_1(n)}x_1, \ldots, b_\ell^{p_\ell(n)}x_\ell\right)_n$$

is equidistributed in the nilmanifold

$$\overline{(b_1^s x_1)}_{s\in\mathbb{R}} \times \cdots \times \overline{(b_\ell^s x_\ell)}_{s\in\mathbb{R}}.$$

(ii) *If $X_i = G_i/\Gamma_i$, $1 \leq i \leq \ell$, are nilmanifolds, then for every $b_i \in G_i$ and $x_i \in X_i$ the sequence*

$$\left(b_1^{[p_1(n)]}x_1, \ldots, b_\ell^{[p_\ell(n)]}x_\ell\right)_n$$

is equidistributed in the nilmanifold

$$\overline{(b_1^n x_1)}_n \times \cdots \times \overline{(b_\ell^n x_\ell)}_n.$$

[6]By this we mean that there exist T-invariant sub-σ-algebras $\mathcal{Z}_{k,i}$, $i \in \mathbb{N}$, of $\mathcal{B}$ such that $\mathcal{Z}_k = \bigcup_{i\in\mathbb{N}} \mathcal{Z}_{k,i}$ and for every $i \in \mathbb{N}$, the factors induced by the σ-algebras $\mathcal{Z}_{k,i}$ are isomorphic to k-step nilsystems.

Remarks. (i) In order to prove Theorem 4.1, we can assume that $X_1 = \ldots = X_\ell = X$.

Indeed, in the general case we consider the nilmanifold $\tilde{X} = X_1 \times \cdots \times X_\ell$. Then $\tilde{X} = \tilde{G}/\tilde{\Gamma}$, where $\tilde{G} = G_1 \times \cdots \times G_\ell$ is connected and simply connected and $\tilde{\Gamma} = \Gamma_1 \times \cdots \times \Gamma_\ell$ is a discrete cocompact subgroup of $\tilde{G}$. Each b_i can be considered as an element of $\tilde{G}$ and each x_i as an element of $\tilde{X}$.

(ii) If $X = G/\Gamma$ is a nilmanifold, since for every $b \in G$ the nil-orbit $(b^n\Gamma)_n$ is equidistributed in $X_b = \overline{\{b^n\Gamma : n \in \mathbb{N}\}}$, using the fact that $b^n g = g(g^{-1}bg)^n$ we have that $(b^n g\Gamma)_n$ is equidistributed in $g \cdot X_{g^{-1}bg}$ (in case G is connected and simply connected a similar formula holds where $n \in \mathbb{N}$ is replaced by $s \in \mathbb{R}$ and the nilmanifold X_b with $Y_b = \overline{\{b^s\Gamma : s \in \mathbb{R}\}}$). Because of this (which is called *change of base point formula*), changing the base point we can assume that $x = \Gamma$ in the previous statement.

The following lemma shows that Part (ii) of Theorem 4.1 follows from Part (i).

Lemma 4.2 (Lemma 5.1, [9]). *Let $\ell \in \mathbb{N}$ and $(a_1(n))_n, \ldots, (a_\ell(n))_n$ be sequences of real numbers. Suppose that for every nilmanifold $X = G/\Gamma$, with G connected and simply connected and every $b_1, \ldots, b_\ell \in G$ the sequence*

$$\left(b_1^{a_1(n)}\Gamma, \ldots, b_\ell^{a_\ell(n)}\Gamma\right)_n$$

is equidistributed in the nilmanifold

$$\overline{(b_1^s\Gamma)}_{s\in\mathbb{R}} \times \cdots \times \overline{(b_\ell^s\Gamma)}_{s\in\mathbb{R}}.$$

Then for every nilmanifold $X = G/\Gamma$, every $b_1, \ldots, b_\ell \in G$ and $x_1, \ldots, x_\ell \in X$ the sequence

$$\left(b_1^{[a_1(n)]}x_1, \ldots, b_\ell^{[a_\ell(n)]}x_\ell\right)_n$$

is equidistributed in the nilmanifold

$$\overline{(b_1^n x_1)}_n \times \cdots \times \overline{(b_\ell^n x_\ell)}_n.$$

Next we give a result needed to prove Part (i) of Theorem 4.1.

Proposition 4.3. *Let $\ell \in \mathbb{N}$ and $p_1, \ldots, p_\ell \in \mathbb{R}[t]$ be strongly independent real polynomials. Let $X = G/\Gamma$ be a nilmanifold with G connected and simply connected and elements $b_i \in G$ acting ergodically on X. Then the sequence*

$$\left(b_1^{p_1(n)}\Gamma, \ldots, b_\ell^{p_\ell(n)}\Gamma\right)_n$$

is equidistributed in the nilmanifold X^ℓ.

Proof. First notice that the sequence $\left(b_1^{p_1(n)}, \ldots, b_\ell^{p_\ell(n)}\right)_n$ is a polynomial sequence in G^ℓ. Indeed, if $p(t) = a_d t^d + \ldots + a_1 t + a_0$, then $b^{p(n)}$ can be written as $(b^{a_d})^{n^d} \cdots (b^{a_1})^n b^{a_0}$. Since $X^\ell = G^\ell/\Gamma^\ell$ with G^ℓ connected and simply connected, we can apply Theorem 3.1, so, in order to prove that $\left(b_1^{p_1(n)}\Gamma, \ldots, b_\ell^{p_\ell(n)}\Gamma\right)_n$ is equidistributed in G^ℓ, it suffices to show that $\left(\pi(b_1^{p_1(n)}\Gamma), \ldots, \pi(b_\ell^{p_\ell(n)}\Gamma)\right)_n$ is equidistributed in Z^ℓ, where $Z = G/([G,G]\Gamma)$ and $\pi : X \to Z$ is the natural projection.

Since G is connected and simply connected, Z is isomorphic to some finite dimensional torus $\mathbb{T}^s$ and every nilrotation in Z is isomorphic to a rotation on $\mathbb{T}^s$. Hence, for every

$1 \leq i \leq \ell$ we have $\pi(b_i\Gamma) = (\beta_{i,1}\mathbb{Z}, \dots, \beta_{i,s}\mathbb{Z})$, where $\beta_{i,j} \in \mathbb{R}$ and $(\beta_{i,1}, \dots, \beta_{i,s})$ is the projection of b_i on $\mathbb{T}^s$ (note that the s is bounded by the dimension of X). Since every b_i acts ergodically on X, we have that for all $1 \leq i \leq \ell$ the set of real numbers $\{1, \beta_{i,1}, \dots, \beta_{i,s}\}$ is rationally independent. Also, for every $t \in \mathbb{R}$ and $1 \leq i \leq \ell$ we have that $\pi(b_i^t\Gamma) = (t\beta'_{i,1}\mathbb{Z}, \dots, t\beta'_{i,s}\mathbb{Z})$, for some $\beta'_{i,j} \in \mathbb{R}$ with $\beta'_{i,j}\mathbb{Z} = \beta_{i,j}\mathbb{Z}$, and so $\pi\Big(b_i^{p_i(n)}\Gamma\Big) = (p_i(n)\beta'_{i,1}\mathbb{Z}, \dots, p_i(n)\beta'_{i,s}\mathbb{Z})$. Note that $\{1, \beta'_{i,1}, \dots, \beta'_{i,s}\}$ is also a rationally independent set for all $1 \leq i \leq \ell$.

Our objective now is to establish the equidistribution of the sequence

$$\big((p_1(n)\beta'_{1,1}\mathbb{Z}, \dots, p_1(n)\beta'_{1,s}\mathbb{Z}, \dots, p_\ell(n)\beta'_{\ell,1}\mathbb{Z}, \dots, p_\ell(n)\beta'_{\ell,s}\mathbb{Z})\big)_n$$

on $\mathbb{T}^{\ell s}$. To verify this we use Weyl's criterion (see [25]; for a reference in English and an extensive study in uniform distribution see also [22]).

Let $\mathbf{h} = (h_{1,1}, \dots, h_{1,s}, \dots, h_{\ell,1}, \dots, h_{\ell,s}) \in \mathbb{Z}^{\ell s} \setminus \{(0, \dots, 0)\}$. Because of the aforementioned rational independence, not all the sums $\sum_{j=1}^{s} h_{i,j}\beta'_{i,j}$, $1 \leq i \leq \ell$, are equal to 0, so, using the strong independence of the family of polynomials $\{p_1, \dots, p_\ell\}$ we get that the polynomial $\sum_{i=1}^{\ell}\Big(\sum_{j=1}^{s} h_{i,j}\beta'_{i,j}\Big)p_i(n)$ has at least one non-constant irrational coefficient. So

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mathrm{e}\left(\mathbf{h} \cdot (p_1(n)\beta'_{1,1}, \dots, p_1(n)\beta'_{1,s}, \dots, p_\ell(n)\beta'_{\ell,1}, \dots, p_\ell(n)\beta'_{\ell,s})\right) =$$

$$\lim_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} e\Big(\sum_{i=1}^{\ell}\Big(\sum_{j=1}^{s} h_{i,j}\beta'_{i,j}\Big)p_i(n)\Big) = 0.$$

By Weyl's equidistribution criterion, the result follows. □

The last ingredient in proving Part (i) of Theorem 4.1 is the following lemma:

Lemma 4.4 (Lemma 5.2, [9]). *Let $X = G/\Gamma$ be a nilmanifold with G connected and simply connected. Then for every $b_1, \dots, b_\ell \in G$ there exists an $s_0 \in \mathbb{R}$ such that for all $1 \leq i \leq \ell$ the element $b_i^{s_0}$ acts ergodically on the nilmanifold $\overline{(b_i^s\Gamma)}_{s\in\mathbb{R}}$.*

We are now ready to prove Theorem 4.1.

Proof of Theorem 4.1. Using Lemma 4.2 we see that Part (ii) of Theorem 4.1 follows from Part (i). To establish Part (i) let $b_1, \dots, b_\ell \in G$. By Lemma 4.4 there exists a non-zero $s_0 \in \mathbb{R}$ such that for every $1 \leq i \leq \ell$ the element $b_i^{s_0}$ acts ergodically on the nilmanifold $\overline{(b_i^s\Gamma)}_{s\in\mathbb{R}}$. Using Proposition 4.3 for the elements $b_i^{s_0}$ and the polynomials $p_i(s)/s_0$ (who trivially are still strongly independent) we get that the sequence $\Big(b_1^{p_1(n)}\Gamma, \dots, b_\ell^{p_\ell(n)}\Gamma\Big)_n$ is equidistributed in the nilmanifold $\overline{(b_1^s\Gamma)}_{s\in\mathbb{R}} \times \cdots \times \overline{(b_\ell^s\Gamma)}_{s\in\mathbb{R}}$ and the conclusion follows. □

5. Proof of main results

In this last section we present the proofs of our main results, namely Theorems 2.1, 2.12 and 2.14.

We first give the proof of Theorem 2.1. In order to do so we recall from [8] that the nilfactor $\mathcal{Z}$ of a system is characteristic for the family $\{p_1, \dots, p_\ell\}$, where $p_1, \dots, p_\ell \in \mathbb{R}[t]$ are strongly independent real polynomials. Actually, the statement in [8] is about *nice* families of polynomials (see definition below), a notion more general than the strong independence than we use here.

Definition ([8])**.** Let $\ell \in \mathbb{N}$ and for $N \in \mathbb{N}$, let $\mathcal{P}_N = \{p_{1,N}, \dots, p_{\ell,N}\}$ be a family of polynomials with real coefficients. We say that the collection $(P_N)_N$ is *nice* if for every $N \in \mathbb{N}$ the polynomials $p_{i,N}$ and $p_{i,N} - p_{j,N}$, $i \neq j$, are non-constant and their leading coefficients are independent of N.

The following lemma shows that for a nice collection of polynomial families the nilfactor is the characteristic factor as well (a different proof of this fact is also given in [24]).

Lemma 5.1 (Lemma 4.7, [8])**.** *Let $(\{p_{1,N}, \dots, p_{\ell,N}\})_N$ be a nice collection of polynomial families, $(X, \mathcal{B}, \mu, T)$ be a system and suppose that one of the functions $f_1, \dots, f_\ell \in L^\infty(\mu)$ is orthogonal to the nilfactor $\mathcal{Z}$. Then for any Følner sequence $(\Phi_N)_N$ in $\mathbb{Z}$* [7] *and any bounded two parameter sequence $(c_{N,n})_{N,n}$ of real numbers we have*

$$\lim_{N\to\infty} \frac{1}{|\Phi_N|} \sum_{n\in\Phi_N} c_{N,n} T^{[p_{1,N}(n)]} f_1 \cdot \ldots \cdot T^{[p_{\ell,N}(n)]} f_\ell = 0, \tag{12}$$

where the convergence takes place in $L^2(\mu)$.

Remark. For $\ell \in \mathbb{N}$, let $\{p_1, \dots, p_\ell\}$ be a strongly independent family of polynomials. Then, trivially this collection is nice, so we have (12), hence the nilfactor $\mathcal{Z}$ is the characteristic factor for this family.

For reasons of complicity, and since the aforementioned lemma is a crucial result for our study, we will present most of the details of the proof in the special setting where $p_{i,N} = p_i$, $c_{N,n} = 1$, and $\Phi_N = \{1, \dots, N\}$ for $N \in \mathbb{N}$ (which is a Følner sequence in $\mathbb{N}$). Note that in this case that the sequences of polynomials are constant the notion of "nice" polynomials coincides with the notion of "essentially distinct" polynomials ([2]).

Sketch of proof of special case of Lemma 5.1. We follow the argument in [8], Lemma 4.7. Without loss of generality we assume that f_1 is orthogonal to $\mathcal{Z}$, $\|f_i\|_\infty \leq 1$ for all $1 \leq i \leq \ell$, and that p_1 is a polynomial of maximum degree in $\mathcal{P} = \{p_1, \dots, p_\ell\}$. Under these assumptions it can be easily shown that in order to get the result, it suffices to show

$$\lim_{N\to\infty} \sup_{\|f_0\|_\infty, \|f_2\|_\infty, \dots, \|f_\ell\|_\infty \leq 1} \frac{1}{N} \sum_{n=1}^{N} \left| \int f_0 \cdot T^{[p_1(n)]} f_1 \cdot \ldots \cdot T^{[p_\ell(n)]} f_\ell \, d\mu \right| = 0. \tag{13}$$

For any family of (finitely many) polynomials, we assign a vector $(d, w_d, \dots, w_1)$, where d is the maximum degree of the polynomials and w_i the number of distinct coefficients of

[7] A *Følner sequence in* $\mathbb{Z}$ is a sequence $(\Phi_n)_n$ of finite sunsets of $\mathbb{Z}$ such that for any $m \in \mathbb{Z}$ we have $\lim_{n\to\infty} \frac{|(\Phi_n + m) \cap \Phi_n|}{|\Phi_n|} = 1$.

polynomials of degree i in the family. We call this vector *type* of the family. We compare vector-types by lexicographical order. We will show (13) by using induction on the type of the polynomial family $\{p_1, \dots, p_\ell\}$.

The case where $d = \deg p_1 = 1$ can be treated as in [8], Proposition 5.3 (or, by using Lemmas 4.11 and 4.13 in [11]).

Let $d = \deg p_1 \geq 2$ and suppose that the statement holds for nice polynomial families of type smaller than $(d, w_d, \dots, w_1)$. Using the Cauchy-Schwarz inequality, (13) will follow if we show

$$\lim_{N\to\infty} \sup_{\|f_0\|_\infty, \|f_2\|_\infty, \dots, \|f_\ell\|_\infty \leq 1} \frac{1}{N}\sum_{n=1}^{N} \left| \int f_0 \cdot T^{[p_1(n)]} f_1 \cdot \ldots \cdot T^{[p_\ell(n)]} f_\ell \, d\mu \right|^2 = 0.$$

So, if $\tilde{T} = T \times T$, $\tilde{f} = f \otimes \bar{f}$ and $\tilde{\mu} = \mu \times \mu$, using Cauchy-Schwarz it suffices to show

$$\lim_{N\to\infty} \sup_{\|\tilde{f}_2\|_\infty, \dots, \|\tilde{f}_\ell\|_\infty \leq 1} \left\| \frac{1}{N}\sum_{n=1}^{N} \tilde{T}^{[p_1(n)]} \tilde{f}_1 \cdot \ldots \cdot \tilde{T}^{[p_\ell(n)]} \tilde{f}_\ell \right\|_{L^2(\tilde{\mu})} = 0. \tag{14}$$

Using a variation of the classical van der Corput lemma (see [Lemma 4.6, [8]]), setting $b(n_1, \dots, n_\ell) = \tilde{T}^{n_1} \tilde{f}_1 \cdot \ldots \cdot \tilde{T}^{n_\ell} \tilde{f}_\ell$, (14) follows if we show that for large enough h we have

$$\lim_{N\to\infty} \sup_{\|\tilde{f}_2\|_\infty, \dots, \|\tilde{f}_\ell\|_\infty \leq 1} \left| \frac{1}{N}\sum_{n=1}^{N} \int b([p_1(n+h)], \dots, [p_\ell(n+h)]) \overline{b([p_1(n)], \dots, [p_\ell(n)])} \, d\tilde{\mu} \right| = 0.$$

We factor out the transformation $\tilde{T}^{[p(n)]}$, where $p = p_{i_0}$ for some $1 \leq i_0 \leq \ell$ is chosen so that for every large h the polynomial subfamily $\mathcal{P}(p, h)$ which is obtained by $\{p_1(n + h) - p(n), \dots, p_\ell(n+h) - p(n), p_1(n) - p(n), \dots, p_\ell(n) - p(n)\}$ after successfully removing the smallest number of polynomials so that the resulting family consists of non-constant, essentially distinct polynomials, has type smaller than $\{p_1(n), \dots, p_\ell(n)\}$ (see [Lemma 4.5, [8]]). We write $[p_i(n + h)] - [p(n)] = [p_i(n + h) - p(n)] + e_{1,i}(h, n)$ and $[p_i(n)] - [p(n)] = [p_i(n) - p(n)] + e_{2,i}(h, n)$, for all $1 \leq i \leq \ell$, where $e_{j,i}(h, n)$ are error terms with values in $\{0, 1\}$. For every fixed h we partition the integers into finite number of sets, that depend only on ℓ, where all the error terms are constant. Therefore, it suffices to show that

$$\begin{aligned} \lim_{N\to\infty} \sup_{\|\tilde{f}_2\|_\infty, \dots, \|\tilde{f}_\ell\|_\infty \leq 1} \frac{1}{N}\sum_{n=1}^{N} \bigg| \int & b([p_1(n+h) - p(n)], \dots, [p_\ell(n+h) - p(n)]) \cdot \\ & \overline{b([p_1(n) - p(n)], \dots, [p_\ell(n) - p(n)])} \, d\tilde{\mu} \bigg| = 0. \end{aligned} \tag{15}$$

If it happens some polynomial p_i to have degree 1 (hence $i \neq 1$), we write $p_i(n + h) = p_i(n) + c(h)$, where $c(h) = p_i(h) - p_i(0) \in \mathbb{R}$. Hence, in this case, we have

$$\tilde{T}^{[p_i(n+h)-p(n)]} \tilde{f}_i \cdot \overline{\tilde{T}^{[p_i(n)-p(n)]} \tilde{f}_i} = \tilde{T}^{[p_i(n)-p(n)]} (\tilde{T}^{c(h)+e(h,n)} \tilde{f}_i \cdot \overline{\tilde{f}_i}),$$

for some error terms $e(h, n) \in \{0, 1\}$. Since the error terms take finitely many values, in order to show (15) we can assume that $e(h, n) = 0$. Therefore, it suffices to show that for large h we have that

$$\lim_{N\to\infty} \sup_{\|\tilde{f}_0\|_\infty,\|\tilde{f}_2\|_\infty,\ldots,\|\tilde{f}_r\|_\infty\leq 1} \frac{1}{N}\sum_{n=1}^{N}\left|\int \tilde{f}_0\cdot \tilde{T}^{[p_1(n+h)-p(n)]}\tilde{f}_1\cdot\prod_{i=2}^{r}\tilde{T}^{[\tilde{p}_{h,i}(n)]}\tilde{f}_i\,d\tilde{\mu}\right| = 0 \tag{16}$$

for some $r\in\mathbb{N}$, where all the polynomials $\tilde{p}_{h,i}$ belong to the family $\mathcal{P}(p,h)$. For sufficiently large h, this family is of essentially distinct polynomials with type less than that of $\mathcal{P}$. Since $p_1(n+h)-p(n)$ is the polynomial of maximum degree in $\mathcal{P}(p,h)$ and since f_1 is orthogonal to $\mathcal{Z}(T)$, we have that $\tilde{f}_1$ is orthogonal to $\mathcal{Z}(\tilde{T})$ and the result now follows by the induction hypothesis. □

Proof of Theorem 2.1. We start by using Lemma 5.1 in order to get that the nilfactor $\mathcal{Z}$ is characteristic for the corresponding multiple ergodic average. Via Theorem 3.2 we can assume without loss of generality that our system is an inverse limit of nilsystems. By a standard approximation argument, we can further assume that our system is a nilsystem.

Let $(X=G/\Gamma,\mathcal{G}/\Gamma,m_X,T_b)$ be a nilsystem, where $b\in G$ is ergodic, and $F_1,\ldots,F_\ell\in L^\infty(m_X)$. Our objective now is to show that if $\{p_1,\ldots,p_\ell\}$ is a strongly independent family of polynomials then

$$\lim_{N\to\infty}\sum_{n=1}^{N}F_1(b^{[p_1(n)]}x)\cdot\ldots\cdot F_\ell(b^{[p_\ell(n)]}x)=\int F_1\,dm_X\cdot\ldots\cdot\int F_\ell\,dm_X \tag{17}$$

where the convergence takes place in $L^2(m_X)$. By density, we can assume that the functions $F_1,\ldots,F_\ell$ are continuous. Then we can apply Theorem 4.1 to the nilmanifold X^ℓ, the nilrotation $\tilde{b}=(b,\ldots,b)\in G^\ell$, the point $\tilde{x}=(x,\ldots,x)\in X^\ell$, and the continuous function $\tilde{F}(x_1,\ldots,x_\ell)=F_1(x_1)\cdot\ldots\cdot F_\ell(x_\ell)$, to get that

$$\lim_{N\to\infty}\sum_{n=1}^{N}\tilde{F}(b^{[p_1(n)]}x,\ldots,b^{[p_\ell(n)]}x)=\int\tilde{F}\,dm_{X^\ell}$$

and this gives the desired limit in (17), completing the proof. □

5.1. **Convergence along Primes.**

We first give the definitions and the main ideas in order to prove Theorems 2.12 and 2.14.

We start by recalling the definition of the *von Mangoldt function*, $\Lambda:\mathbb{N}\to\mathbb{R}$, where $\Lambda(n)=\begin{cases}\log(p) & \text{, if } n=p^k \text{ for some } p\in\mathbb{P} \text{ and some } k\in\mathbb{N}\\ 0 & \text{, elsewhere}\end{cases}.$

As in [14] and [21] it is more natural for us to work instead of Λ with the function $\Lambda':\mathbb{N}\to\mathbb{R}$, where $\Lambda'(n)=\mathbf{1}_{\mathbb{P}}(n)\cdot\Lambda(n)=\mathbf{1}_{\mathbb{P}}(n)\cdot\log(n)$.

The function Λ', according to the following lemma, will allow us to relate averages along primes with weighted averages over the integers.

Lemma 5.2 ([13]). *If $a:\mathbb{N}\to\mathbb{C}$ is bounded, then*

$$\lim_{N\to\infty}\left|\frac{1}{\pi(N)}\sum_{p\in\mathbb{P}\cap[1,N]}a(p)-\frac{1}{N}\sum_{n=1}^{N}\Lambda'(n)\cdot a(n)\right|=0.$$

For $w > 2$, let

$$W = \prod_{p\in\mathbb{P}\cap[1,w-1]} p$$

be the product of primes bounded above by w. For $r \in \mathbb{N}$, let

$$\Lambda'_{w,r}(n) = \frac{\phi(W)}{W} \cdot \Lambda'(Wn + r),$$

where ϕ is the Euler function, be the *modified von Mangoldt function*.

Definition. For $\ell \in \mathbb{N}$, we call the setting $(X, \mathcal{B}, \mu, T_1, \ldots, T_\ell)$ a *system*, where $T_1, \ldots, T_\ell \colon X \to X$ are invertible commuting measure preserving transformations on the probability space $(X, \mathcal{B}, \mu)$.

The proposition below, the proof of which relies on a deep result due to Green and Tao ([18]) on the inverse conjecture for the Gowers norms, will provide us with a crucial intermediate step in order to prove Theorems 2.12 and 2.14, as well as Theorems 2.18 and 2.19 (we will actually use a very weak version of it for all these results).

Proposition 5.3 (Proposition 3.2, [21])**.** *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{B}, \mu, T_1, \ldots, T_m)$ be a system, $p_{i,j} \in \mathbb{R}[t]$ be real polynomials, $1 \le i \le m$, $1 \le j \le \ell$ and $f_1, \ldots, f_\ell \in L^\infty(\mu)$.*
Then,

$$\max_{1\le r\le W,\ (r,W)=1} \left\| \frac{1}{N}\sum_{n=1}^{N} (\Lambda'_{w,r}(n) - 1) \cdot (\prod_{i=1}^{m} T_i^{[p_{i,1}(Wn+r)]}) f_1 \cdot \ldots \cdot (\prod_{i=1}^{m} T_i^{[p_{i,\ell}(Wn+r)]}) f_\ell \right\|_{L^2(\mu)}$$

converges to 0 *as* $N \to \infty$ *and then* $w \to \infty$.

Proof of Theorem 2.12. We borrow the arguments from the proof of Theorem 1.3 from [14] (see also Theorem 1.3 in [21]). By Lemma 5.2 it suffices to show that the sequence

$$A(N) := \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) \cdot T^{[q(n)]} f_1 \cdot T^{2[q(n)]} f_2 \cdot \ldots \cdot T^{\ell[q(n)]} f_\ell$$

converges in $L^2(\mu)$ to the same limit as the sequence $\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{\ell n} f_\ell$ as $N \to \infty$. For w (which gives a corresponding W), $r \in \mathbb{N}$, we define

$$B_{w,r}(N) := \frac{1}{N}\sum_{n=1}^{N} T^{[q(Wn+r)]} f_1 \cdot T^{2[q(Wn+r)]} f_2 \cdot \ldots \cdot T^{\ell[q(Wn+r)]} f_\ell.$$

For any $\varepsilon > 0$, using Proposition 5.3 with $m = \ell$, $T_i = T$, $1 \le i \le \ell$ and $p_{i,j} = \begin{cases} 0 & , \text{ if } i \le \ell - j \\ q & , \text{ elsewhere} \end{cases}$, for sufficiently large N and some w_0 we have

$$\left\| A(W_0 N) - \frac{1}{\phi(W_0)} \sum_{1\le r\le W_0,\ (r,W_0)=1} B_{w_0,r}(N) \right\|_{L^2(\mu)} < \varepsilon.$$

Note at this point that for all $W, r \in \mathbb{N}$ we have that $q(Wt + r) \notin c\mathbb{Q}[t] + d$ for $c, d \in \mathbb{R}$, for otherwise q would have the same property contradicting our assumption.

By Theorem 2.10, we have that for any $1 \leq r \leq W_0$ the sequence $(B_{w_0,r}(N))_N$ converges to the same limit as the sequence $\frac{1}{N}\sum_{n=1}^{N} T^n f_1 \cdot T^{2n} f_2 \cdot \ldots \cdot T^{\ell n} f_\ell$, and since

$$\lim_{N\to\infty} \|A(W_0N+r) - A(W_0N)\|_{L^2(\mu)} = 0$$

for every $1 \leq r \leq W_0$, we get the result. □

Proof of Theorem 2.14. The proof is analogous to the previous one. In this case we define $A(N) := \frac{1}{N}\sum_{n=1}^{N} \Lambda'(n) \cdot T^{[p_1(n)]}f_1 \cdot \ldots \cdot T^{[p_\ell(n)]}f_\ell$ and for $w, r \in \mathbb{N}$, $B_{w,r}(N) := \frac{1}{N}\sum_{n=1}^{N} T^{[p_1(Wn+r)]}f_1 \cdot \ldots \cdot T^{[p_\ell(Wn+r)]}f_\ell$. We use Proposition 5.3 with $m = \ell$, $T_i = T$, $1 \leq i \leq \ell$, $p_{i,j} = \begin{cases} 0 & , \text{ if } i \neq j \\ p_i & , \text{ if } i = j \end{cases}$ and we note that the family $\{\tilde{p}_1, \ldots, \tilde{p}_\ell\}$, where $\tilde{p}_i(t) = p_i(Wt+r)$, is strongly independent for all $W, r \in \mathbb{N}$. (Indeed, if for some $(\lambda_1, \ldots, \lambda_\ell) \in \mathbb{R}^\ell \setminus \{\vec{0}\}$, $d \in \mathbb{R}$, $q \in \mathbb{Q}[t]$ and $W, r \in \mathbb{N}$ we had $\sum_{i=1}^{\ell} \lambda_i p_i(Wt+r) = q(t) + d$, then $\sum_{i=1}^{\ell} \lambda_i p_i(t) = \tilde{q}(t) + d$, where $\tilde{q}(t) = q((t-r)/W) \in \mathbb{Q}[t]$, a contradiction to the strong independence assumption.) The result now follows similarly to the previous proof since by Theorem 2.1, we have that for any $1 \leq r \leq W_0$ the sequence $(B_{w_0,r}(N))_N$ converges, in $L^2(\mu)$, to $\prod_{i=1}^{\ell} \int f_i \, d\mu$. □

5.2. **Recurrence along shifted primes.** In this last subsection we prove Theorems 2.18, 2.19, 2.20 and 2.21. In order to show Theorem 2.18 we use Theorem 2.1 (iii) from [5].

Theorem 5.4 (Theorem 2.1, [5])**.** *Let $\ell \in \mathbb{N}$ and $(X, \mathcal{B}, \mu, T)$ be a system. Then for every $A \in \mathcal{B}$ with $\mu(A) > 0$ there exists a constant $c \equiv c_{\ell,\mu(A)} > 0$ such that*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu\left(A \cap T^{-n}A \cap T^{-2n}A \cap \ldots \cap T^{-\ell n}A\right) \geq c. \tag{18}$$

Remark. Actually, the limit in (18) exists by [19].

Proof of Theorem 2.18. Using Proposition 5.3 with $m = \ell$, $T_i = T$, $1 \leq i \leq \ell$, $r = 1$ and $p_{i,j}(n) = \begin{cases} 0 & , \text{ if } i \leq \ell - j \\ p(n-1) & , \text{ elsewhere} \end{cases}$ and combining Theorem 5.4 with Theorem 2.10 we have, for sufficiently large $\omega \in \mathbb{N}$, that

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \Lambda'_{\omega,1}(n) \cdot \mu\left(A \cap T^{-[p(Wn)]}A \cap T^{-2[p(Wn)]}A \cap \ldots \cap T^{-\ell[p(Wn)]}A\right) > 0,$$

from which we get the required non-empty intersection with $\mathbb{P} - 1$. □

Proof of Theorem 2.19. The proof is analogous to the one of Theorem 2.18. More specifically, we use Theorem 2.10 instead of Theorem 5.4 and Proposition 5.3 with $m=\ell$, $T_i=T$, $1\leq i\leq \ell$, $r=1$ and $p_{i,j}(n)=\begin{cases} 0 & \text{, if } i\neq j \\ p_i(n-1) & \text{, if } i=j \end{cases}$ to get, for some sufficiently large $\omega\in\mathbb{N}$, that

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Lambda'_{\omega,1}(n)\cdot\mu\left(A\cap T^{-[p_1(Wn)]}A\cap\ldots\cap T^{-[p_\ell(Wn)]}A\right)>0,$$

from which we get the result. □

Proof of Theorems 2.20 and 2.21. Follow immediately by using Theorems 2.18 and 2.19 together with Furstenberg's correspondence principle. □

Remarks. (1) According to Lemma 5.2, we have that the conclusions of Theorems 2.18, 2.19, and so of Theorems 2.20 and 2.21, are satisfied for a set of integers n with positive relative density in the shifted primes $\mathbb{P}-1$ (the analogous results, by a similar argument, hold for the set $\mathbb{P}+1$ as well).

(2) In the special case where the polynomials $\{p_1,\ldots,p_\ell\}$ satisfy $p_i(0)=1/2$, for all $1\leq i\leq \ell$, the results of Theorems 2.18, 2.19, 2.20 and 2.21 follow by Theorem 1.2 in [21].

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