EXTENDED RAMSEY THEORY FOR WORDS REPRESENTING RATIONALS

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Abstract. Ramsey theory for words over a finite alphabet was unified in the work of Carlson, who also presented a method to extend the theory for words over an infinite alphabet, but subject to a fixed dominating principle. In the present work we establish an extension of Carlson's approach to countable ordinals and Schreier-type families developing an extended Ramsey theory for dominated words over a doubly infinite alphabet (in fact for ω-$\mathbb{Z}^*$-located words), and we apply this theory, exploiting the Budak-Işik-Pym representation of rational numbers, to obtain an analogous partition theory for the set of rational numbers.

1. Introduction

We introduce (in Definition 2.1) the notion of ω-$\mathbb{Z}^*$-located words ($\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$) over an alphabet $\Sigma = \{\alpha_n : \ n \in \mathbb{Z}^*\}$ dominated by a two-sided sequence $\vec{k} = (k_n)_{n\in\mathbb{Z}^*}$ of natural numbers to be a located word $w = w_{n_1} \dots w_{n_l}$ over Σ with domain $\{n_1 < \dots < n_l\} \subseteq \mathbb{Z}^*$ and in addition for $1 \leq i \leq l$, $w_{n_i} \in \{\alpha_1, \dots, \alpha_{k_{n_i}}\}$ if $n_i \in \mathbb{N}$ and $w_{n_i} \in \{\alpha_{-k_{n_i}}, \dots, \alpha_{-1}\}$ if $-n_i \in \mathbb{N}$. The inspiration for this notion came from the representation of rational numbers introduced by T. Budak, N. Işik and J. Pym in [BIP] (Theorem 4.2), according to which a rational number q has a unique representation as

$$q = \sum_{s=1}^{\infty} q_{-s} \frac{(-1)^s}{(s+1)!} \ + \ \sum_{r=1}^{\infty} q_r (-1)^{r+1} r!$$

where $(q_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N} \cup \{0\}$ with $0 \leq q_{-s} \leq s$ for every $s > 0$, $0 \leq q_r \leq r$ for every $r > 0$ and $q_{-s} = q_r = 0$ for all but finite many r, s. So, the set of non-zero rational numbers can be identified with the set of all the ω-$\mathbb{Z}^*$-located words over the alphabet $\Sigma = \{\alpha_n : \ n \in \mathbb{Z}^*\}$, where $\alpha_{-n} = \alpha_n = n$ for $n \in \mathbb{N}$, dominated by the sequence $(k_n)_{n\in\mathbb{Z}^*}$, where $k_{-n} = k_n = n$ for $n \in \mathbb{N}$.

The whole of infinitary Ramsey theory can be obtained for ω-$\mathbb{Z}^*$-located words; indeed, The classical Ramsey theory for ω-$\mathbb{Z}^*$-located words consisting of a partition theorem for the family of m-tuples of ω-$\mathbb{Z}^*$-located

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words with m a natural number and a partition theorem for the family of infinite sequences of ω-$\mathbb{Z}^*$-located words considering suitable partition sets, as Borel sets with regard to the product topology, is presented in Section 2 (Theorems 2.5 and 2.6 respectively) and on the main follows from the fundamental work of Carlson in [C] (Theorem 2.3 below).

In Section 3, we extend this classical Ramsey theory to partitions involving ξ-Schreier-type sequences of ω-$\mathbb{Z}^*$-located words for every countable ordinal ξ (Definition 3.2), which constitute the natural transfinite analogues of the m-tuples of ω-$\mathbb{Z}^*$-located words (with m a natural number). The basic feature that diffentiates the families of ξ-Schreier-type sequences of ω-$\mathbb{Z}^*$-located words from each other is their complexity, as measured by a suitable Cantor-Bendixson type index introduced in Definition 3.15 (Proposition 3.18). Thus Theorem 3.5, a partition theorem for the family of ξ-Schreier sequences of ω-$\mathbb{Z}^*$-located words for each countable ordinal ξ, constitutes an extension of Theorem 2.6 (corresponding to ordinal level $\xi = m$, a natural number).

The main result of this Section 3, and indeed of the paper, is Theorem 3.21, which, on the one hand strengthens Theorem 3.5, in case the set of all finite sequences of variable ω-$\mathbb{Z}^*$-located words is participated by a family $\mathcal{F}$ which is a tree, and on the other hand implies a stronger countable ordinal form of Theorem 2.5 in case the partition sets are clopen in the product topology. More specifically, Theorem 3.5 provides no information on whether the ξ-homogeneous family, for a countable ordinal ξ, falls in $\mathcal{F}$ or in its complement, while Theorem 3.21 provides a criterion on how to decide, in terms of a suitable Cantor-Bendixson type index of $\mathcal{F}$: if this index is greater than $\xi + 1$ the ξ-homogeneous family falls in $\mathcal{F}$ and if less than ξ in its complement.

The set of non-zero rational numbers with addition, using the representation of rational numbers given by Budak-Işik-Pym, can be identified with the set of all the ω-$\mathbb{Z}^*$-located words over the alphabet $\Sigma = \{\alpha_n : \ n \in \mathbb{Z}^*\}$, where $\alpha_{-n} = \alpha_n = n$ for $n \in \mathbb{N}$ dominated by the sequence $(k_n)_{n\in\mathbb{Z}^*}$ where $k_{-n} = k_n = n$ for $n \in \mathbb{N}$, via the function $g : \widetilde{L}(\Sigma, \vec{k}) \to \mathbb{Q} \setminus \{0\}$, which sends a word $w = q_{t_1} \dots q_{t_l} \in \widetilde{L}(\Sigma, \vec{k})$ to the rational number

$$q(w) = \textstyle\sum_{t\in dom^-(w)} q_t \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t\in dom^+(w)} q_t (-1)^{t+1} t!,$$

since g is one-to-one and onto and $g(w_1 \star w_2) = g(w_1) + g(w_2)$ for every $w_1 <_{R_1} w_2 \in \widetilde{L}(\Sigma \cup \{0\}, \vec{k})$. Applying the results of Sections 2 and 3 via the function g, to the rationals with addition, we obtain, in Section 4, an analogous Ramsey theory for the rational numbers, starting from the partition

Theorem 4.1, a strengthened van der Waerden theorem for the set of rational numbers. Analogous partition theorems can be obtained for semigroups representable as ω-$\mathbb{Z}^*$-located words.

Notation. Let $\mathbb{N}=\{1,2,\dots\}$ be the set of natural numbers, $\mathbb{Z}=\{\dots,-2,-1,0,1,2,\dots\}$ be the set of integer numbers, $\mathbb{Z}^-=\{n\in\mathbb{Z}: n<0\}$, $\mathbb{Z}^*=\mathbb{Z}\setminus\{0\}$ and $\mathbb{Q}=\{\frac{m}{n}:\ m\in\mathbb{Z},\ n\in\mathbb{N}\}$ be the of rational numbers. For a non-empty set X we denote by $[X]^{<\omega}$ the set of all the finite subsets of X and by $[X]^{<\omega}_{>0}$ the set of all the non-empty finite subsets of X.

2. Classical Ramsey theory for ω-$\mathbb{Z}^*$-located words

The purpose of this section is to introduce the ω-$\mathbb{Z}^*$-located words over an alphabet $\Sigma=\{\alpha_n:\ n\in\mathbb{Z}^*\}$ dominated by a two-sided sequence $\vec{k}=(k_n)_{n\in\mathbb{Z}^*}$ of natural numbers (Definition 2.1) and to develop the classical Ramsey theory for such words. These results on the main follow from the partition Theorem 2.3 for ω-located words (Definition 2.2) proved by Carlson in [C]. Let start with the necessary terminology and notation.

Definition 2.1. Let an alphabet $\Sigma=\{\alpha_n:\ n\in\mathbb{Z}^*\}$, a sequence $\vec{k}=(k_n)_{n\in\mathbb{Z}^*}\subseteq\mathbb{N}$ such that $k_n\leq k_{n+1}$ and $k_{-n}\leq k_{-(n+1)}$ for every $n\in\mathbb{N}$ and let $\upsilon\notin\Sigma$ an entity which is called a *variable*. We will call these assumptions "standard assumptions" throughout the paper.
An *ω-$\mathbb{Z}^*$-located word* over the alphabet Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{Z}^*$ into the alphabet Σ such that $w(n)=w_n\in\{\alpha_1,\dots,\alpha_{k_n}\}$ for every $n\in F\cap\mathbb{N}$ and $w_n\in\{\alpha_{-k_n},\dots,\alpha_{-1}\}$ for every $n\in F\cap\mathbb{Z}^-$.
A *variable ω-$\mathbb{Z}^*$-located word* over the alphabet Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{Z}^*$ into $\Sigma\cup\{\upsilon\}$ such that $w(n)=w_n\in\{\upsilon,\alpha_1,\dots,\alpha_{k_n}\}$ for every $n\in F\cap\mathbb{N}$ and $w_n\in\{\upsilon,\alpha_{-k_n},\dots,\alpha_{-1}\}$ for every $n\in F\cap\mathbb{Z}^-$ and there exist $n_1\in F\cap\mathbb{N}$ and $n_2\in F\cap\mathbb{Z}^-$ such that $w_{n_1}=w_{n_2}=\upsilon$.

So, the set $\widetilde{L}(\Sigma,\vec{k})$ of all (constant) ω-$\mathbb{Z}^*$-located words over Σ dominated by $\vec{k}$ is:

$$\widetilde{L}(\Sigma,\vec{k})=\{w=w_{n_1}\dots w_{n_l}: l\in\mathbb{N}, n_1<\dots<n_l\in\mathbb{Z}^* \text{ and } w_{n_i}\in\{\alpha_1,\dots,\ \alpha_{k_{n_i}}\} \text{ if } n_i>0,\ w_{n_i}\in\{\alpha_{-k_{n_i}},\dots,\alpha_{-1}\} \text{ if } n_i<0 \text{ for every } 1\leq i\leq l\},$$

and the set of *variable ω-$\mathbb{Z}^*$-located words* over Σ dominated by $\vec{k}$ is:

$\widetilde{L}(\Sigma,\vec{k};\upsilon) = \{w = w_{n_1}\ldots w_{n_l} : l \in \mathbb{N}, n_1 < \ldots < n_l \in \mathbb{Z}^*, w_{n_i} \in \{\upsilon,\alpha_1,\ldots,\alpha_{k_{n_i}}\}$ if $n_i > 0$, $w_{n_i} \in \{\upsilon,\alpha_{-k_{n_i}},\ldots,\alpha_{-1}\}$ if $n_i < 0$ for all $1 \leq i \leq l$ and there exist $n_1 < 0 < n_2$ with $w_{n_1} = w_{n_2} = \upsilon\}$.

We set $\widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k}) = \widetilde{L}(\Sigma,\vec{k}) \cup \widetilde{L}(\Sigma,\vec{k};\upsilon)$.
For $w = w_{n_1}\ldots w_{n_l} \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$ the set $dom(w) = \{n_1 < \ldots < n_l\}$ is the *domain* of w. Let $dom^-(w) = \{n \in dom(w) : \ n < 0\}$ and $dom^+(w) = \{n \in dom(w) : \ n > 0\}$.

For $w = w_{n_1}\ldots w_{n_r}, u = u_{m_1}\ldots u_{m_l} \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$ with $dom(w) \cap dom(u) = \emptyset$ we define the concatenating word located on the union of the domains of w, u as

$$w \star u = z_{q_1}\ldots z_{q_{r+l}} \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k}),$$

where $\{q_1 < \ldots < q_{r+l}\} = dom(w) \cup dom(u)$, $z_i = w_i$ if $i \in dom(w)$ and $z_i = u_i$ if $i \in dom(u)$.

We endow the set $\widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$ with a relation $<_{R_1}$ defining for $w, u \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$

$$w <_{R_1} u \iff \begin{array}{l} dom(u) = A_1 \cup A_2 \text{ with } A_1, A_2 \neq \emptyset \text{ such that} \\ \max A_1 < \min dom(w) \leq \max dom(w) < \min A_2. \end{array}$$

We define

$$\widetilde{L}^\infty(\Sigma,\vec{k};\upsilon) = \{(w_n)_{n\in\mathbb{N}} \subseteq \widetilde{L}(\Sigma,\vec{k};\upsilon) : \ w_n <_{R_1} w_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

For $m \in \mathbb{N}$ we set

$$\widetilde{L}^m(\Sigma,\vec{k};\upsilon) = \{(w_1,\ldots,w_m) : w_1 <_{R_1} \ldots <_{R_1} w_m \in \widetilde{L}(\Sigma,\vec{k};\upsilon)\}.$$

For every $(p,q) \in \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ we define the functions

$$T_{(p,q)} : \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k}) \longrightarrow \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$$

setting, for $w = w_{n_1}\ldots w_{n_l} \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$, $T_{(0,0)}(w) = w$ and, for $(p,q) \in \mathbb{N}\times\mathbb{N}$, $T_{(p,q)}(w) = u_{n_1}\ldots u_{n_l}$, where, for $1 \leq i \leq l$
$u_{n_i} = w_{n_i}$ if $w_{n_i} \in \Sigma$,
$u_{n_i} = \alpha_p$ if $w_{n_i} = \upsilon$, $n_i > 0$ and $p \leq k_{n_i}$,
$u_{n_i} = \alpha_{k_{n_i}}$ if $w_{n_i} = \upsilon$, $n_i > 0$ and $p > k_{n_i}$,
$u_{n_i} = \alpha_{-q}$ if $w_{n_i} = \upsilon$, $n_i < 0$ and $q \leq k_{n_i}$, and
$u_{n_i} = \alpha_{-k_{n_i}}$ if $w_{n_i} = \upsilon$, $n_i < 0$ and $q > k_{n_i}$.
We remark that for every $(p,q) \in \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ we have $dom(T_{(p,q)}(w)) = dom(w)$ for $w \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$, $T_{(p,q)}(w) = w$ for $w \in \widetilde{L}(\Sigma,\vec{k})$ and $T_{(p,q)}(w\star u) = T_{(p,q)}(w)\star T_{(p,q)}(u)$ for every $w, u \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$ with $dom(w)\cap dom(u) = \emptyset$. Also, $T_{(p,q)}(\widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})) \subseteq \widetilde{L}(\Sigma,\vec{k})$ for every $(p,q) \in \mathbb{N}\times\mathbb{N}$.

Extracted ω-$\mathbb{Z}^*$-located words, Extractions. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. We fix a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$.

An *extracted variable ω-$\mathbb{Z}^*$-located word* of $\vec{w}$ is a variable ω-$\mathbb{Z}^*$-located word $u \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$ such that

$$u = T_{(p_1,q_1)}(w_{n_1}) \star \ldots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \ldots < n_\lambda \in \mathbb{N}$, $(p_i, q_i) \in \mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ with $0 \leq p_i \leq k_{n_i}$, $0 \leq q_i \leq k_{-n_i}$ for every $1 \leq i \leq \lambda$ and $(0,0) \in \{(p_1, q_1), \ldots, (p_\lambda, q_\lambda)\}$. The set of all the extracted variable ω-$\mathbb{Z}^*$-located words of $\vec{w}$ is denoted by $\widetilde{EV}(\vec{w})$.

An *extracted ω-$\mathbb{Z}^*$-located word* of $\vec{w}$ is an ω-$\mathbb{Z}^*$-located word $z \in \widetilde{L}(\Sigma, \vec{k})$ with

$$z = T_{(p_1,q_1)}(w_{n_1}) \star \ldots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \ldots < n_\lambda \in \mathbb{N}$ and $(p_i, q_i) \in \mathbb{N} \times \mathbb{N}$ with $1 \leq p_i \leq k_{n_i}$, $1 \leq q_i \leq k_{-n_i}$ for every $1 \leq i \leq \lambda$. The set of all the extracted ω-$\mathbb{Z}^*$-located words of $\vec{w}$ is denoted by $\widetilde{E}(\vec{w})$. Let

$$\widetilde{EV}^{\infty}(\vec{w}) = \{\vec{u} = (u_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) : u_n \in \widetilde{EV}(\vec{w}) \text{ for every } n \in \mathbb{N}\}.$$

For $m \in \mathbb{N}$ we set
$\widetilde{EV}^{m}(\vec{w}) = \{(u_1, \ldots, u_m) : u_1 <_{R_1} \ldots <_{R_1} u_m \in \widetilde{EV}(\vec{w})\}$.
If $\vec{u} \in \widetilde{EV}^{\infty}(\vec{w})$, then we say that $\vec{u}$ is an *extraction* of $\vec{w}$ and we write $\vec{u} \prec \vec{w}$. Notice that for $\vec{u}, \vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ we have $\vec{u} \prec \vec{w}$ if and only if $\widetilde{EV}(\vec{u}) \subseteq \widetilde{EV}(\vec{w})$.

Definition 2.2. Let an alphabet $\Sigma = \{\alpha_n : n \in \mathbb{N}\}$, a sequence $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $k_n \leq k_{n+1}$ for every $n \in \mathbb{N}$ and let $\upsilon \notin \Sigma$ a *variable*. An *ω–located word* over the alphabet Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{N}$ into the alphabet Σ such that $w(n) = w_n \in \{\alpha_1, \ldots, \alpha_{k_n}\}$ for every $n \in F \cap \mathbb{N}$.
A *variable ω–located word* over the alphabet Σ dominated by $\vec{k}$ is a function w from a non-empty, finite subset F of $\mathbb{N}$ into $\Sigma \cup \{\upsilon\}$ such that $w(n) = w_n \in \{\upsilon, \alpha_1, \ldots, \alpha_{k_n}\}$ for every $n \in F \cap \mathbb{N}$ and there exist $n_1 \in F \cap \mathbb{N}$ such that $w_{n_1} = \upsilon$.

So, the set of ω-located words over Σ dominated by $\vec{k}$ is

$L(\Sigma, \vec{k}) = \{w = w_{n_1} \ldots w_{n_l} : l \in \mathbb{N}, n_1 < \ldots < n_l \in \mathbb{N}, w_{n_i} \in \{\alpha_1, \ldots, \alpha_{k_{n_i}}\}$ for every $1 \leq i \leq l\}$,

and the set of variable ω-located words over Σ dominated by $\vec{k}$ respectively, is

$L(\Sigma, \vec{k}; \upsilon) = \{w = w_{n_1} \ldots w_{n_l} : l \in \mathbb{N}, n_1 < \ldots < n_l \in \mathbb{N}, w_{n_i} \in \{\upsilon, \alpha_1, \ldots, \alpha_{k_{n_i}}\}$ for every $1 \leq i \leq l$ and there exists $1 \leq i \leq l$ with $w_{n_i} = \upsilon\}$.

Let $L(\Sigma \cup \{\upsilon\}, \vec{k}) = L(\Sigma, \vec{k}) \cup L(\Sigma, \vec{k}; \upsilon)$.

For $w, u \in L(\Sigma \cup \{\upsilon\}, \vec{k})$ we write

$$w <_{R_2} u \iff \max dom(w) < \min dom(u).$$

We define

$$L^\infty(\Sigma, \vec{k}; \upsilon) = \{(w_n)_{n\in\mathbb{N}} \subseteq L(\Sigma, \vec{k}; \upsilon) : w_n <_{R_2} w_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

For every $p \in \mathbb{N} \cup \{0\}$ are defined the functions

$$T_p : L(\Sigma \cup \{\upsilon\}, \vec{k}) \to L(\Sigma \cup \{\upsilon\}, \vec{k})$$

setting for $w = w_{n_1} \dots w_{n_l} \in L(\Sigma \cup \{\upsilon\}, \vec{k})$: $T_0(w) = w$ and, for $p \in \mathbb{N}$, $T_p(w) = u_{n_1} \dots u_{n_l}$, where, for $1 \leq i \leq l$, $u_{n_i} = w_{n_i}$ if $w_{n_i} \in \Sigma$, $u_{n_i} = \alpha_p$ if $w_{n_i} = \upsilon$ and $p \leq k_{n_i}$ and finally $u_{n_i} = \alpha_{k_{n_i}}$ if $w_{n_i} = \upsilon$ and $p > k_{n_i}$.
Let $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$. The set $EV(\vec{w})$ of all the *extracted variable ω-located words* of $\vec{w}$ contains the words of the form

$$T_{p_1}(w_{n_1}) \star \dots \star T_{p_\lambda}(w_{n_\lambda}),$$

where $\lambda \in \mathbb{N}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$ and $p_1, \dots, p_\lambda \in \mathbb{N} \cup \{0\}$ such that $0 \leq p_i \leq k_{n_i}$ for every $1 \leq i \leq \lambda$ and $0 \in \{p_1, \dots, p_\lambda\}$, and the set $E(\vec{w})$ of all the *extracted ω-located words* of $\vec{w}$ contains the words of the form $T_{p_1}(w_{n_1}) \star \dots \star T_{p_\lambda}(w_{n_\lambda})$, where $\lambda \in \mathbb{N}$, $n_1 < \dots < n_\lambda \in \mathbb{N}$ and $p_1, \dots, p_\lambda \in \mathbb{N}$ such that $1 \leq p_i \leq k_{n_i}$ for every $1 \leq i \leq \lambda$. Let

$$EV^\infty(\vec{w}) = \{\vec{u} = (u_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon) : u_n \in EV(\vec{w}) \text{ for every } n \in \mathbb{N}\}.$$

If $\vec{u} \in EV^\infty(\vec{w})$, then we say that $\vec{u}$ is an *extraction* of $\vec{w}$ and we write $\vec{u} \prec \vec{w}$. Notice that $\vec{u} \prec \vec{w}$ if and only if $EV(\vec{u}) \subseteq EV(\vec{w})$.

With the previous terminology we can state the following fundamental partition theorem of Carlson for infinite sequences of variable ω-located words, which is a corollary of the much stronger Theorem 15 of [C]. According to Theorem 15, the partition families $A_1, \dots, A_r$, referred in the statement of Theorem 2.3 below, can be members of a wider class of sets, but we restrict to the class of Borel sets, since it is a sufficiently large, universally understood class.

Theorem 2.3 (Carlson, [C])**.** *Let $\Sigma = \{\alpha_n : n \in \mathbb{N}\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable, $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence and $r \in \mathbb{N}$. If $L^\infty(\Sigma, \vec{k}; \upsilon) = A_1 \cup \dots \cup A_r$ where A_i is a Borel set (with regard to the product topology on sequences of elements of $L(\Sigma, \vec{k}; \upsilon)$, where $L(\Sigma, \vec{k}; \upsilon)$ has the discrete topology) for all $i = 1, \dots, r$, then there exists a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ and $1 \leq i_0 \leq r$ such that*

$$EV^\infty(\vec{w}) \in A_{i_0}.$$

For the sake of completeness, we will state the following partition theorem for variable ω-located words. A proof of this theorem can be given by the stronger Theorem 2.3, but it can be proved independently either similarly to Lemma 5.9 in [C], as referred in [C], or as Theorem 1.4 in [F4].

Theorem 2.4 (Carlson, [C]). *Let $\Sigma = \{\alpha_n : n \in \mathbb{N}\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable, $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence and $r, s \in \mathbb{N}$. If $L(\Sigma, \vec{k}; \upsilon) = A_1 \cup \ldots \cup A_r$ and $L(\Sigma, \vec{k}) = C_1 \cup \ldots \cup C_s$, then there exists a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ and $1 \le i_0 \le r$, $1 \le j_0 \le s$ such that*

$$EV(\vec{w}) \in A_{i_0} \text{ and } E(\vec{w}) \in C_{j_0}.$$

Now we will state and prove a partition theorem for infinite sequences of variable ω-$\mathbb{Z}^*$-located words using Theorem 2.3.

Theorem 2.5. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and let $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, $r \in \mathbb{N}$. If $\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) = A_1 \cup \ldots \cup A_r$ where A_i is a Borel set (with regard to the product topology on sequences of elements of $\widetilde{L}(\Sigma, \vec{k}; \upsilon)$, where $\widetilde{L}(\Sigma, \vec{k}; \upsilon)$ has the discrete topology) for all $i = 1, \ldots, r$, then there exists an extraction $\vec{u} = (u_n)_{n\in\mathbb{N}}$ of $\vec{w}$ and $1 \le i_0 \le r$, such that*

$$\widetilde{EV}^\infty(\vec{u}) \subseteq A_{i_0}.$$

Proof. We will order the set $\mathbb{N}\times\mathbb{N}$. For $(p, q) \in \mathbb{N}\times\mathbb{N}$ we set $i(p, q)$ to be the least $n \in \mathbb{N}$ such that $p \le k_n$ and $q \le k_{-n}$ and then we define the order $<_*$ of $\mathbb{N}\times\mathbb{N}$ so that $(p_1, q_1) <_* (p_2, q_2)$ if and only if either $i(p_1, q_1) < i(p_2, q_2)$ or both $i(p_1, q_1) = i(p_2, q_2)$ and (p_1, q_1) is less that (p_2, q_2) in the lexicographical ordering (i.e. either $p_1 < p_2$ or both $p_1 = p_2$ and $q_1 < q_2$).

Let $\mathbb{N} \times \mathbb{N} = \{\beta_1 <_* \beta_2 <_* \beta_3 <_* \ldots\}$ and let the increasing sequence $\vec{l} = (l_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ such that $\beta_{l_n} = (k_n, k_{-n})$. We set $\widetilde{\Sigma} = \{\beta_n : n \in \mathbb{N}\}$ and we define the function $h : L(\widetilde{\Sigma}, \vec{l}; \upsilon) \to \widetilde{EV}(\vec{w})$ which sends $t_{n_1} \ldots t_{n_\lambda} \in L(\widetilde{\Sigma}, \vec{l}; \upsilon)$ to

$$h(t_{n_1} \ldots t_{n_\lambda}) = T_{(p_1,q_1)}(w_{n_1}) \star \ldots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}),$$

where for $1 \le i \le \lambda$, $(p_i, q_i) = (0, 0)$ if $t_{n_i} = \upsilon$ and $(p_i, q_i) = (\mu_1, \mu_2)$ if $t_{n_i} = \beta_\mu = (\mu_1, \mu_2)$. The function h is one-to-one and onto $\widetilde{EV}(\vec{w})$.
Now, we define an extension $\overline{h}$ of h to $L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon)$ setting

$$\overline{h} : L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon) \to \widetilde{EV}^\infty(\vec{w}) \text{ with } \overline{h}((t_n)_{n\in\mathbb{N}}) = (h(t_n))_{n\in\mathbb{N}}$$

for every $(t_n)_{n\in\mathbb{N}} \in L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon)$. Note that $\overline{h}$ is a homeomorphism with respect to the product topologies.

According to Theorem 2.3, there exist a sequence $\vec{s} = (s_n)_{n\in\mathbb{N}} \in L^\infty(\widetilde{\Sigma}, \vec{l}; \upsilon)$ and $1 \le i_0 \le r$, such that $EV^\infty(\vec{s}) \subseteq (\overline{h})^{-1}(A_{i_0})$. Set $u_n = h(s_n) \in \widetilde{EV}(\vec{w})$ for every $n \in \mathbb{N}$ and $\vec{u} = (u_n)_{n\in\mathbb{N}}$. Then $\vec{u} = (u_n)_{n\in\mathbb{N}} \prec \vec{w}$ and $\widetilde{EV}^\infty(\vec{u}) \subseteq \overline{h}(EV^\infty(\vec{s})) \subseteq A_{i_0}$. □

Theorem 2.5 implies the following partition theorem for ordered m-tuples of variable ω-$\mathbb{Z}^*$-located words for every natural number m.

Theorem 2.6. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and let $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, r, $m \in \mathbb{N}$. If $\widetilde{L}^m(\Sigma, \vec{k}; \upsilon) = C_1 \cup \ldots \cup C_r$, then there exists an extraction $\vec{u} = (u_n)_{n\in\mathbb{N}}$ of $\vec{w}$ and $1 \leq i_0 \leq r$, such that*

$$\widetilde{EV}^m(\vec{u}) \subseteq C_{i_0}.$$

Proof. Set $A_i = \{(x_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) : \ (x_1, \ldots, x_m) \in C_i\}$ for $i = 1, \ldots, r$. Then $\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) = A_1 \cup \ldots \cup A_r$, and $A_1, \ldots, A_r$ are Borel subsets of $\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. The conclusion follows from Theorem 2.5. □

Remark 2.7. 1) The initial case ($m = 1$) of Theorem 2.6 has a proof independent of Theorem 2.5, applying Theorem 2.4, via the function $h : L(\widetilde{\Sigma}, \vec{l}; \upsilon) \to \widetilde{EV}(\vec{w})$ defined in the proof of Theorem 2.5.

2) Theorem 2.6 can be proved by induction from its initial case for $m = 1$, using Lemma 3.6 as in the proof of Theorem 3.5.

3) Theorems 2.5 and 2.6 imply the analogous partition theorems for constant ω-$\mathbb{Z}^*$-located words.

3. Ramsey theoretic results involving Schreier systems for ω-$\mathbb{Z}^*$-located words

The starting point of this section is Theorem 3.5, where we state and prove an extended to every countable order Ramsey-type partition theorem ([R]) for variable ω-$\mathbb{Z}^*$-located words over an alphabet $\Sigma = \{\alpha_n : \ n \in \mathbb{Z}^*\}$, dominated by a sequence $\vec{k} = (k_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$. It is an extension to every countable order ξ of Theorem 2.5 corresponding to the case $\xi = m$, a natural number. As consequences of Theorem 3.5 we can get an extended to every countable order Ramsey-type partition theorem for ω-located words (see Corollary 3.8).

The main result of this section is Theorem 3.21. This theorem is a strengthening of Theorem 3.5 in case the partition family $\mathcal{F}$ is a tree, providing a criterion, in terms of a Cantor-Bandixson type index on $\mathcal{F}$, on how to decide whether the ξ-homogeneous family falls in $\mathcal{F}$ or in its complement. We note, in Theorem 3.24 and Remark 3.25, that Theorem 3.21 can be considered as a strengthening of the particular case of Theorem 2.5, in which the partition families of $\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ are clopen in the product topology.

The vehicle for proving these extended Ramsey type partition Theorems 3.5 and 3.21 is the Schreier systems $(\widetilde{L}^\xi(\Sigma, \vec{k}))_{\xi<\omega_1}$ and $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon))_{\xi<\omega_1}$, consisting of families of finite orderly sequences of (constant and variable respectively) ω-$\mathbb{Z}^*$-located words over the alphabet Σ dominated by the sequence $\vec{k}$ (given in Definition 3.2). Instrumental for this definition are the Schreier sets $\mathcal{A}_\xi$, consisting of finite subsets of $\mathbb{N}$, which are defined below in Definition 3.1 (employing (in case 3(iii)) the Cantor normal form of ordinals (cf. [KM], [L])). Schreier sets were systematically studied in [F1] and [F3].

Notation. For $s_1, s_2 \in [\mathbb{N}]^{<\omega}_{>0}$ we write $s_1 < s_2 \iff \max s_1 < \min s_2$.

Definition 3.1 (The Schreier systems, [F1, Def. 7], [F2, Def. 1.5], [F3, Def. 1.4])**.** For every non-zero, countable, limit ordinal λ choose and fix a strictly increasing sequence $(\lambda_n)_{n\in\mathbb{N}}$ of successor ordinals smaller than λ with $\sup_n \lambda_n = \lambda$ (i.e. λ is the least ordinal such that $\lambda_n \leq \lambda$ for all $n \in \mathbb{N}$). The system $(\mathcal{A}_\xi)_{\xi<\omega_1}$ is defined recursively as follows:

(1) $\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$;

(2) $\mathcal{A}_{\zeta+1} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in \mathcal{A}_\zeta\}$;

(3i) $\mathcal{A}_{\omega^{\beta+1}} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \bigcup_{i=1}^n s_i$, where $n = \min s_1$, $s_1 < \ldots < s_n$ and $s_1, \ldots, s_n \in \mathcal{A}_{\omega^\beta}\}$

(3ii) for a non-zero, countable limit ordinal λ,
$\mathcal{A}_{\omega^\lambda} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s \in \mathcal{A}_{\omega^{\lambda_n}}$ with $n = \min s\}$; and

(3iii) for a limit ordinal ξ such that $\omega^\alpha < \xi < \omega^{\alpha+1}$ for some $0 < \alpha < \omega_1$, if $\xi = \omega^\alpha p + \sum_{i=1}^m \omega^{a_i} p_i$, where $m \in \mathbb{N}$ with $m \geq 0$, $p, p_1, \ldots, p_m$ are natural numbers with $p, p_1, \ldots, p_m \geq 1$ (so that either $p > 1$, or $p = 1$ and $m \geq 1$) and $a, a_1, \ldots, a_m$ are ordinals with $a > a_1 > \ldots > a_m > 0$,
$\mathcal{A}_\xi = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = s_0 \cup (\bigcup_{i=1}^m s_i)$ with $s_m < \ldots < s_1 < s_0$, $s_0 = s_1^0 \cup \ldots \cup s_p^0$ with $s_1^0 < \ldots < s_p^0 \in \mathcal{A}_{\omega^a}$, and $s_i = s_1^i \cup \ldots \cup s_{p_i}^i$ with $s_1^i < \ldots < s_{p_i}^i \in \mathcal{A}_{\omega^{a_i}}$ for every $1 \leq i \leq m\}$.

The Schreier systems are special systems of Ramsey families defined in [F3].

A *system of Ramsey families* is a collection $\mathcal{A} = (\mathcal{A}_\xi)_{\xi<\omega_1}$ of finite subsets of $\mathbb{N}$ defined recursively, by fixing for every non-zero, limit countable ordinal ξ a strictly increasing sequence of successor ordinals smaller than ξ with $\sup_n \xi_n = \xi$, as follows:

$\mathcal{A}_0 = \{\emptyset\}$ and $\mathcal{A}_1 = \{\{n\} : n \in \mathbb{N}\}$;

for every countable ordinal ζ

$\mathcal{A}_{\zeta+1} = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s = \{n\} \cup s_1$, where $n \in \mathbb{N}$, $\{n\} < s_1$ and $s_1 \in \mathcal{A}_\zeta\}$;

and for every non-zero, limit countable ordinal ξ

$\mathcal{A}_\xi = \{s \in [\mathbb{N}]^{<\omega}_{>0} : s \in \mathcal{A}_{\xi_n}$ with $n = \min s\}$.

With suitable choices of sequences $(\xi_n)_{n\in\mathbb{N}}$ for each countable, non-zero limit ordinal ξ one can define interesting systems of Ramsey families. The Schreier systems are the simplest systems of Ramsey families defined by employing the Cantor normal form of the countable ordinals. We note that a Schreier system is a purely combinatorial entity, it nevertheless arose gradually in connection with the theory of Banach spaces (more details can be found in the introduction of [FN2]).

We point out that the results in this section can be stated for systems of Ramsey families in stead of Schreier systems and also that they do not depend on the particular choice of the converging sequences, as the complexity of a family $\mathcal{A}_\xi$, as measured by its Cantor-Bendixson index, is independent of the particular choice of the converging sequences.

We will define now the Schreier systems of ω-$\mathbb{Z}^*$-located words.

Notation. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions (see Drefinition 2.1). We define the *finite orderly sequences of ω-$\mathbb{Z}^*$-located words* over Σ dominated by $\vec{k}$ as follows:

$\widetilde{L}^{<\infty}(\Sigma,\vec{k}) = \{\mathbf{w} = (w_1,\dots,w_l) : l \in \mathbb{N}, w_1 <_{R_1} \dots <_{R_1} w_l \in \widetilde{L}(\Sigma,\vec{k})\} \cup \{\emptyset\}$,

$\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) = \{\mathbf{w} = (w_1,\dots,w_l) : l \in \mathbb{N}, w_1 <_{R_1} \dots <_{R_1} w_l \in \widetilde{L}(\Sigma,\vec{k};\upsilon)\} \cup \{\emptyset\}$,

and $\widetilde{L}^{<\infty}(\Sigma\cup\{\upsilon\},\vec{k}) = \widetilde{L}^{<\infty}(\Sigma,\vec{k}) \cup \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$.

Definition 3.2 (The Schreier systems $(\widetilde{L}^{\xi}(\Sigma,\vec{k}))_{\xi<\omega_1}$ and $(\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon))_{\xi<\omega_1}$)**.**
We define

$$\widetilde{L}^{0}(\Sigma,\vec{k}) = \{\emptyset\} = \widetilde{L}^{0}(\Sigma,\vec{k};\upsilon),\ \textit{and for every countable ordinal}\ \xi \geq 1,$$

$$\widetilde{L}^{\xi}(\Sigma,\vec{k}) = \{(w_1,\dots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k}) : \{\min dom^+(w_1),\dots,\min dom^+(w_l)\} \in \mathcal{A}_\xi\},$$

$$\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon) = \{(w_1,\dots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) : \{\min dom^+(w_1),\dots,\min dom^+(w_l)\} \in \mathcal{A}_\xi\}.$$

Remark 3.3. (i) $\emptyset \notin \widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)$ for every $\xi \geq 1$.

(ii) $\widetilde{L}^{m}(\Sigma,\vec{k};\upsilon) = \{(w_1,\dots,w_m) : w_1 <_{R_1} \dots <_{R_1} w_m \in \widetilde{L}(\Sigma,\vec{k};\upsilon)\}$ *for* $m \in \mathbb{N}$.

(iii) $\widetilde{L}^{\omega}(\Sigma,\vec{k};\upsilon) = \{(w_1,\dots,w_n) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) : n \in \mathbb{N},$ *and* $\min dom^+(w_1) = n\}$.

(iv) Alternatively we could define the sets $\widetilde{L}^{\xi}(\Sigma,\vec{k})$, $\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)$ via the negative part of the domain of the words as follows

$$\widetilde{L}^{\xi}(\Sigma,\vec{k}) = \{(w_1,\dots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k}) : \{|\max dom^-(w_1)|,\dots,|\max dom^-(w_l)|\} \in \mathcal{A}_\xi\},$$

$$\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon) = \{(w_1,\dots,w_l) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon) : \{|\max dom^-(w_1)|,\dots,|\max dom^-(w_l)|\} \in \mathcal{A}_\xi\}.$$

The following proposition justifies the recursiveness of the systems $(\widetilde{L}^{\xi}(\Sigma,\vec{k}))_{\xi<\omega_1}$ and $(\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon))_{\xi<\omega_1}$.
For a family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma\cup\{\upsilon\},\vec{k})$ and $t \in \widetilde{L}(\Sigma\cup\{\upsilon\},\vec{k})$, we set

$$\mathcal{F}(t) = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma \cup \{\upsilon\}, \vec{k}) : \text{either } \mathbf{w} = (w_1, \dots, w_l) \neq \emptyset \text{ and } (t, w_1, \dots, w_l) \in \mathcal{F} \text{ or } \mathbf{w} = \emptyset \text{ and } (t) \in \mathcal{F}\},$$
$$\mathcal{F} - t = \{\mathbf{w} \in \mathcal{F} : \text{either } \mathbf{w} = (w_1, \dots, w_l) \neq \emptyset \text{ and } t <_{R_1} w_1, \text{ or } \mathbf{w} = \emptyset\}.$$

Proposition 3.4. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. For each countable ordinal $\xi \geq 1$, there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of countable ordinals with $\xi_n < \xi$ such that for $s \in \widetilde{L}(\Sigma, \vec{k})$ and $t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$, with $\min dom^+(s) = \min dom^+(t) = n$, we have*
$$\widetilde{L}^{\xi}(\Sigma, \vec{k})(s) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}) \cap (\widetilde{L}^{<\infty}(\Sigma, \vec{k}) - s), \quad \textit{and}$$
$$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap (\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) - t).$$

Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$, and $(\xi_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\sup_n \xi_n = \xi$ if ξ is a limit ordinal.

Proof. It follows from Theorem 1.6 in [F3], according to which for each countable ordinal $\xi > 0$ there exists a sequence $(\xi_n)_{n\in\mathbb{N}}$ of countable ordinals with $\xi_n < \xi$, such that $\mathcal{A}_\xi(n) = \mathcal{A}_{\xi_n} \cap [\{n+1, n+2, \dots\}]^{<\omega}$ for every $n \in \mathbb{N}$, where, $\mathcal{A}_\xi(n) = \{s \in [\mathbb{N}]^{<\omega} : s \in [\mathbb{N}]^{<\omega}_{>0}, n < \min s \text{ and } \{n\} \cup s \in \mathcal{A}_\xi \text{ or } s = \emptyset$ and $\{n\} \in \mathcal{A}_\xi\}$. Moreover, $\xi_n = \zeta$ for every $n \in \mathbb{N}$ if $\xi = \zeta + 1$, and $(\xi_n)_{n\in\mathbb{N}}$ is a strictly increasing sequence with $\sup_n \xi_n = \xi$ if ξ is a limit ordinal. □

In order to state the following Ramsey type partition theorem on Schreier families for variable ω-$\mathbb{Z}^*$-located words, we need the following notation:

Notation. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. For $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, $\mathbf{w} = (w_1, \dots, w_l) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and $t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$, we set:
$\widetilde{EV}^{<\infty}(\vec{w}) = \{\mathbf{u} = (u_1, \dots, u_l) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : \ l \in \mathbb{N}, u_1, \dots, u_l \in \widetilde{EV}(\vec{w})\} \cup \{\emptyset\}$,
$\widetilde{EV}(\mathbf{w}) = \{T_{(p_1,q_1)}(w_{n_1}) \star \dots \star T_{(p_\lambda,q_\lambda)}(w_{n_\lambda}) \in \widetilde{L}(\Sigma, \vec{k}; \upsilon) : \ 1 \leq n_1 < \dots < n_\lambda \leq l$ and $(p_i, q_i) \in (\mathbb{N} \times \mathbb{N}) \cup \{(0, 0)\}$ with $0 \leq p_i \leq k_{n_i}, \ 0 \leq q_i \leq k_{-n_i}$ for every $1 \leq i \leq \lambda$ and $(0, 0) \in \{(p_1, q_1), \dots, (p_\lambda, q_\lambda)\}\}$, and
$\widetilde{EV}^{<\infty}(\mathbf{w}) = \{\mathbf{u} = (u_1, \dots, u_l) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : l \in \mathbb{N}, u_1, \dots, u_l \in \widetilde{EV}(\mathbf{w})\} \cup \{\emptyset\}$.

Observe that the set $\widetilde{EV}(\mathbf{w})$ is finite. Also, we set

$\vec{w} - t = (w_n)_{n \geq l} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, where $l = \min\{n \in \mathbb{N} : t <_{R_1} w_n\}$,

$\vec{w} - \mathbf{w} = \vec{w} - w_l$,

$\mathbf{w} - t = (w_n, \dots, w_l)$ for $n = \min\{1 \leq i \leq l : \ t <_{R_1} w_i\}$ if $\{1 \leq i \leq l : t <_{R_1} w_i\} \neq \emptyset$ and $\mathbf{w} - t = \emptyset$ otherwise.

Theorem 3.5 (Ramsey type partition theorem on Schreier families for variable ω-$\mathbb{Z}^*$-located words)**.** *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. For* *every countable ordinal $\xi \geq 1$, every family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and every infinite orderly sequence $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ of variable ω-$\mathbb{Z}^*$-located words there exists a variable extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ such that:*

either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,

or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.

A proof of this theorem can be given by the stronger Theorem 2.5, as the partition family $\mathcal{F}$ of $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ can be extended to a partition family A_1 of $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ which is clopen (and consequently Borel) in the product topology. But, in view of Proposition 3.4 on the recursiveness of a Schreier system, we provide, for the sake of completeness a proof of this theorem by induction, starting from the initial case (m=1) of Theorem 2.6, which, as we have mentioned (in Remark 2.7), has a proof independent of Theorem 2.5.

In the proof of this partition theorem we will make use of a diagonal argument, contained in the following lemma.

Lemma 3.6. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\vec{w} = (w_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, and*

$$\Pi = \{(t, \vec{s}) : t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon), \vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon) \text{ with } \vec{s} \prec \vec{w} \text{ and } t <_{R_1} s_1\}.$$

If a subset $\mathcal{R}$ of Π satisfies

(i) *for every $(t, \vec{s}) \in \Pi$, there exists $(t, \vec{s}_1) \in \mathcal{R}$ with $\vec{s}_1 \prec \vec{s}$; and*

(ii) *for every $(t, \vec{s}) \in \mathcal{R}$ and $\vec{s}_1 \prec \vec{s}$, we have $(t, \vec{s}_1) \in \mathcal{R}$,*

then there exists $\vec{u} \prec \vec{w}$, such that $(t, \vec{s}) \in \mathcal{R}$ for all $t \in \widetilde{EV}(\vec{u})$ and $\vec{s} \prec \vec{u} - t$.

Proof. Let $u_0 = w_1$. According to condition (i), there exists $\vec{s}_1 = (s^1_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ with $\vec{s}_1 \prec \vec{w} - u_0$ such that $(u_0, \vec{s}_1) \in \mathcal{R}$. Let $u_1 = s^1_1$. Then $u_0 <_{R_1} u_1$ and $u_0, u_1 \in \widetilde{EV}(\vec{w})$. We assume now that there have been constructed $\vec{s}_1, \ldots, \vec{s}_n \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ and $u_0, u_1, \ldots, u_n \in \widetilde{EV}(\vec{w})$, with $\vec{s}_n \prec \ldots \prec \vec{s}_1 \prec \vec{w}$, $u_0 <_{R_1} u_1 <_{R_1} \ldots <_{R_1} u_n$ and $(t, \vec{s}_i) \in \mathcal{R}$ for all $t \in \widetilde{EV}((u_0, \ldots, u_{i-1}))$ and $1 \leq i \leq n$.

We will construct $\vec{s}_{n+1}$ and u_{n+1}. Let $\{t_1, \ldots, t_l\} = \widetilde{EV}((u_0, \ldots, u_n))$. According to condition (i), there exists $\vec{s}^{\,1}_{n+1}, \ldots, \vec{s}^{\,l}_{n+1} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ such that $\vec{s}^{\,l}_{n+1} \prec \ldots \prec \vec{s}^{\,1}_{n+1} \prec \vec{s}_n - u_n$ and $(t_i, \vec{s}^{\,i}_{n+1}) \in \mathcal{R}$ for every $1 \leq i \leq l$. Set $\vec{s}_{n+1} = \vec{s}^{\,l}_{n+1}$. If $\vec{s}_{n+1} = (s^{n+1}_m)_{m\in\mathbb{N}}$, set $u_{n+1} = s^{n+1}_1$. Of course $u_n <_{R_1} u_{n+1}$, $u_{n+1} \in \widetilde{EV}(\vec{w})$ and, according to condition (ii), $(t_i, \vec{s}_{n+1}) \in \mathcal{R}$ for all $1 \leq i \leq l$.

Set $\vec{u} = (u_0, u_1, u_2, \ldots) \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. Then $\vec{u} \prec \vec{w}$, since $u_0 <_{R_1} u_1 <_{R_1} \ldots \in \widetilde{EV}(\vec{w})$. Let $t \in \widetilde{EV}(\vec{u})$ and $\vec{s} \prec \vec{u} - t$. Set $n_0 = \min\{n \in \mathbb{N} : \ t \in \widetilde{EV}((u_0, u_1, \ldots, u_n))\}$. Since $t \in \widetilde{EV}((u_0, u_1, \ldots, u_{n_0}))$, we have $(t, \vec{s}_{n_0+1}) \in$

$\mathcal{R}$. Then, according to (ii), we have $(t, \vec{u} - u_{n_0}) \in \mathcal{R}$, since $\vec{u} - u_{n_0} \prec \vec{s}_{n_0+1}$, and also $(t, \vec{s}) \in \mathcal{R}$, since $\vec{s} \prec \vec{u} - u_{n_0} = \vec{u} - t$. □

We will now prove Theorem 3.5.

Proof of Theorem 3.5. Let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ and $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. For $\xi = 1$ the theorem is valid, according to Theorem 2.5. Let $\xi > 1$. Assume that the theorem is valid for every $\zeta < \xi$. Let $t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon)$ with $\min dom^+(t) = n$ and $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ with $\vec{s} \prec \vec{w}$ and $t <_{R_1} s_1$. According to Proposition 3.4, there exists $\xi_n < \xi$ such that

$$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap (\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) - t).$$

Using the induction hypothesis, there exists $\vec{s}_1 \prec \vec{s}$ such that

either $\widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}_1) \subseteq \mathcal{F}(t)$,

or $\widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}_1) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)$.

Then $\vec{s}_1 \prec \vec{s} \prec \vec{w}$, and

either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{s}_1) \subseteq \mathcal{F}(t)$,

or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{s}_1) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)$.

Let $\mathcal{R}_1 = \{(t, \vec{s}) :\ t \in \widetilde{L}(\Sigma, \vec{k}; \upsilon),\ \vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon), \vec{s} \prec \vec{w},$
$t <_{R_1} s_1$, and either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{s}) \subseteq \mathcal{F}(t)$
or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{s}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)\}$.

The family $\mathcal{R}_1$ satisfies the conditions (i) (by the above arguments) and (ii) (obviously) of Lemma 3.6. Hence, there exists $\vec{w}_1 = (w_n^1)_{n\in\mathbb{N}} \prec \vec{w}$ such that $(t, \vec{s}) \in \mathcal{R}_1$ for all $t \in \widetilde{EV}(\vec{w}_1)$ and $\vec{s} \prec \vec{w}_1 - t$.

Let $\mathcal{F}_1 = \{t \in \widetilde{EV}(\vec{w}_1) :\ \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{w}_1 - t) \subseteq \mathcal{F}(t)\}$.

We use the induction hypothesis for $\xi = 1$ (Theorem 2.5). There exists a variable extraction $\vec{u} \prec \vec{w}_1$ of $\vec{w}_1$ such that:

either $\widetilde{EV}(\vec{u}) \subseteq \mathcal{F}_1$, or $\widetilde{EV}(\vec{u}) \subseteq \widetilde{L}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_1$.

Since $\vec{u} \prec \vec{w}_1$ we have that $\widetilde{EV}(\vec{u}) \subseteq \widetilde{EV}(\vec{w}_1)$. Thus

either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{u} - t) \subseteq \mathcal{F}(t)$ for all $t \in \widetilde{EV}(\vec{u})$,

or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)(t) \cap \widetilde{EV}^{<\infty}(\vec{u} - t) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}(t)$ for all $t \in \widetilde{EV}(\vec{u})$.

Hence,

either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,

or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$. □

Remark 3.7. (1) The particular case $\xi = m \in \mathbb{N}$ of Theorem 3.5 coincides with Theorem 2.6.

(2) In the particular case $\xi = \omega$, Theorem 3.5 takes the form: if $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) = A_1 \cup \ldots \cup A_r$, $r \in \mathbb{N}$ and $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$, then there exists an extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ and $1 \leq i_0 \leq r$, such that the set $\{(z_1, \ldots, z_n) \in$

$\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)) : n \in \mathbb{N}, \min dom^{+}(z_1) = n$ and $z_1,\dots,z_n \in \widetilde{EV}(\vec{u})\}$ is contained in A_{i_0}.

(3) In analogy to Theorem 3.5 can be proved a Ramsey type partition theorem on Schreier families for (constant) ω-$\mathbb{Z}^*$-located words.

As a consequence of Theorem 3.5 we will prove a Ramsey type partition theorem on Schreier families for variable ω-located words.

Notation. Let $\Sigma = \{\alpha_1, \alpha_2, \dots\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable and $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence. We define the *finite orderly sequences of variable ω-located words* over Σ dominated by $\vec{k}$ as follows:

$$L^{<\infty}(\Sigma,\vec{k};\upsilon) = \{\mathbf{w} = (w_1,\dots,w_l) : l \in \mathbb{N}, w_1 <_{R_2} \dots <_{R_2} w_l \in L(\Sigma,\vec{k};\upsilon)\} \cup \{\emptyset\}.$$

For every countable ordinal $\xi \geq 1$, we set

$$L^{\xi}(\Sigma,\vec{k};\upsilon) = \{(w_1,\dots,w_l) \in L^{<\infty}(\Sigma,\vec{k};\upsilon) : \{\min dom(w_1),\dots,\min dom(w_l)\} \in \mathcal{A}_{\xi}\}.$$

For $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^{\infty}(\Sigma,\vec{k};\upsilon)$ we set:

$$EV^{<\infty}(\vec{w}) = \{\mathbf{u} = (u_1,\dots,u_l) \in L^{<\infty}(\Sigma,\vec{k};\upsilon) : l \in \mathbb{N}, u_1,\dots,u_l \in EV(\vec{w})\} \cup \{\emptyset\}.$$

Corollary 3.8 (Ramsey type partition theorem on Schreier families for variable ω-located words). *Let $\Sigma = \{\alpha_1, \alpha_2, \dots\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable, $\vec{k} = (k_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence and $\xi \geq 1$ a countable ordinal. For a partition family $\mathcal{F} \subseteq L^{<\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{w} \in L^{\infty}(\Sigma,\vec{k};\upsilon)$, there exists an extraction $\vec{u} \prec \vec{w}$ of $\vec{w}$ such that:*

either $L^{\xi}(\Sigma,\vec{k};\upsilon) \cap EV^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,

or $L^{\xi}(\Sigma,\vec{k};\upsilon) \cap EV^{<\infty}(\vec{u}) \subseteq L^{<\infty}(\Sigma,\vec{k};\upsilon) \setminus \mathcal{F}$.

Proof. We set $\widetilde{\Sigma} = \{\alpha_n : n \in \mathbb{Z}^*\}$, $\vec{k}_* = (\widetilde{k}_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$ and $\vec{w}_* = (\widetilde{w}_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\widetilde{\Sigma},\vec{k}_*;\upsilon)$ where $\alpha_{-n} = \alpha_n$, $\widetilde{k}_{-n} = \widetilde{k}_n = k_n$ and $\widetilde{w}_n = \upsilon_{-n} \star w_n$ for every $n \in \mathbb{N}$. Let $\varphi : \widetilde{L}(\widetilde{\Sigma},\vec{k}_*;\upsilon) \to L(\Sigma,\vec{k};\upsilon)$ with $\varphi(w_{n_1}\dots w_{n_l}) = w_{n_{i_0}}\dots w_{n_l}$, where $n_{i_0} = \min dom^{+}(w_{n_1}\dots w_{n_l})$ and $\widetilde{\varphi} : \widetilde{L}^{<\infty}(\widetilde{\Sigma},\vec{k}_*;\upsilon) \to L^{<\infty}(\Sigma,\vec{k};\upsilon)$ with $\widetilde{\varphi}(u_1,\dots,u_l) = (\varphi(u_1),\dots,\varphi(u_l))$. Then, we apply Theorem 3.5 for the family $\widetilde{\varphi}^{-1}(\mathcal{F})$ and the sequence $\vec{w}_*$. □

In order to prove Theorem 3.21, a strengthening of Theorem 3.5 in case the partition family $\mathcal{F}$ is a tree, we will prove three basic properties of the Schreier families of variable ω-$\mathbb{Z}^*$-located words (Propositions 3.10, 3.11 and 3.18 below).

Let start with the necessary notations and definitions.

Definition 3.9. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$.

(i) $\mathcal{F}$ is *thin* if there are no elements $\mathbf{w}, \mathbf{u} \in \mathcal{F}$ with $\mathbf{w} \propto \mathbf{u}$ and $\mathbf{w} \neq \mathbf{u}$.
(ii) $\mathcal{F}^* = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : \ \mathbf{w} \propto \mathbf{u}$ for some $\mathbf{u} \in \mathcal{F}\} \cup \{\emptyset\}$.
(iii) $\mathcal{F}$ is a *tree* if $\mathcal{F}^* = \mathcal{F}$.
(iv) $\mathcal{F}_* = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) : \ \mathbf{w} \in \widetilde{EV}^{<\infty}(\mathbf{u})$ for some $\mathbf{u} \in \mathcal{F}\} \cup \{\emptyset\}$.
(v) $\mathcal{F}$ is *hereditary* if $\mathcal{F}_* = \mathcal{F}$.

Proposition 3.10. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. Every family $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$, for $\xi < \omega_1$ is thin.*

Proof. It follows from the fact that the families $\mathcal{A}_\xi$ are thin (cf. [F3])(which means that if $s, t \in \mathcal{A}_\xi$ and $s \propto t$, then $s = t$). □

Proposition 3.11. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and $\xi \geq 1$ a countable ordinal. Then*

(i) *every infinite orderly sequence $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ has canonical representation with respect to $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$, which means that there exists a unique strictly increasing sequence $(m_n)_{n\in\mathbb{N}}$ in $\mathbb{N}$ so that $(s_1, \ldots, s_{m_1}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ and $(s_{m_{n-1}+1}, \ldots, s_{m_n}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ for every $n > 1$; and,*
(ii) *every non-empty finite orderly sequence $\mathbf{s} = (s_1, \ldots, s_k) \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ has canonical representation with respect to $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$, so either $\mathbf{s} \in (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ or there exists unique $n \in \mathbb{N}$, and $m_1, \ldots, m_n \in \mathbb{N}$ with $m_1 < \ldots < m_n \leq k$ such that either $(s_1, \ldots, s_{m_1})$, $\ldots, (s_{m_{n-1}+1}, \ldots, s_{m_n}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ and $m_n = k$, or $(s_1, \ldots, s_{m_1}), \ldots,$ $(s_{m_{n-1}+1}, \ldots, s_{m_n}) \in \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$, $(s_{m_n+1}, \ldots, s_k) \in (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$.*

Proof. It follows from the fact that every non-empty increasing sequence (finite or infinite) in $\mathbb{N}$ has canonical representation with respect to $\mathcal{A}_\xi$ (cf. [F3]) and that the family $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ is thin (Proposition 3.10). □

Now, using Proposition 3.11, we can give an alternative description of the second horn of the dichotomy described in Theorem 3.5 in case the partition family is a tree.

Proposition 3.12. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\xi \geq 1$ a countable ordinal and $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be a tree. Then*

$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$ *if and only if*
$\mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$.

Proof. Let $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$ and $\mathbf{s} = (s_1, \ldots, s_k) \in \mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{u})$. Then $\mathbf{s}$ has canonical representation with respect to $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$

(Proposition 3.11), hence either $\mathbf{s} \in (\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))^* \setminus \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$, as required, or there exists $\mathbf{s}_1 \in \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$ such that $\mathbf{s}_1 \propto \mathbf{s}$. The second case is impossible. Indeed, since $\mathcal{F}$ is a tree and $\mathbf{s} \in \mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{u})$, we have $\mathbf{s}_1 \in \mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{u}) \cap \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$; a contradiction to our assumption. Hence, $\mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon))^* \setminus \widetilde{L}^\xi(\Sigma,\vec{k};\upsilon)$. □

Definition 3.13. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. Identifying every $\mathbf{s} \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ and every $\vec{s} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon))$ with their characteristic functions $x_{r(\mathbf{s})} \in \{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ and $x_{r(\vec{s})} \in \{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ respectively (where $r(\mathbf{s}) = \{s_1,\ldots,s_k\}$ the range of $\mathbf{s} = (s_1,\ldots,s_k) \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$, $r(\vec{s}) = \{s_n : n \in \mathbb{N}\}$ the range of $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ and $r(\emptyset) = \emptyset$), we say that a family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is *pointwise closed* if and only if the family $\{x_{r(\mathbf{s})} : \mathbf{s} \in \mathcal{F}\}$ is closed in the product topology (equivalently by the pointwise convergence topology) of $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ and in analogy a family $\mathcal{U} \subseteq \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ is *pointwise closed* if and only if $\{x_{r(\vec{s})} : \vec{s} \in \mathcal{U}\}$ is closed in $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ with the product topology.

Proposition 3.14. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions.*

(i) *If $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is a tree, then $\mathcal{F}$ is pointwise closed if and only if there does not exist an infinite sequence $(\mathbf{s}_n)_{n\in\mathbb{N}}$ in $\mathcal{F}$ such that $\mathbf{s}_n \propto \mathbf{s}_{n+1}$ and $\mathbf{s}_n \neq \mathbf{s}_{n+1}$ for all $n \in \mathbb{N}$.*

(ii) *If $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is hereditary, then $\mathcal{F}$ is pointwise closed if and only if there does not exist $\vec{s} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ such that $\widetilde{EV}^{<\infty}(\vec{s}) \subseteq \mathcal{F}$.*

(iii) *The hereditary family $(\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}))_*$ is pointwise closed for every countable ordinal ξ and $\vec{u} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$.*

Proof. The proof of (i) follows from the fact that the set $\widetilde{EV}^{<\infty}(\mathbf{s})$ is finite for every $\mathbf{s} \in \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ and the proof of (ii) follows from (i). The statement (iii) can be proved by induction on ξ, using (ii). The main idea of the proof is that given $\vec{s} = (s_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ with $\widetilde{EV}^{<\infty}(\vec{s}) \subseteq (\widetilde{L}^\xi(\Sigma,\vec{k};\upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}))_*$, then, according to the pigeonhole principle, there exists $k \leq \min dom^+(s_1)$ such that $(s_2,\ldots,s_n) \in (\widetilde{L}^{\xi_k}(\Sigma,\vec{k};\upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}))_*$ for every $n \in \mathbb{N}$ (using Proposition 3.4). □

Let $\vec{s} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$. For a hereditary and pointwise closed family $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ we will define the strong Cantor-Bendixson index $sO_{\vec{s}}(\mathcal{F})$ of $\mathcal{F}$ with respect to $\vec{s}$.

Definition 3.15. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\vec{s} \in \widetilde{L}^\infty(\Sigma,\vec{k};\upsilon)$ and let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ be a hereditary and pointwise closed family. For every $\xi < \omega_1$ we define the families $(\mathcal{F})^\xi_{\vec{s}}$ inductively as follows:

We define $(\mathcal{F})^0_{\vec{s}} = \mathcal{F}$.
For every $\mathbf{w} = (w_1, \dots, w_l) \in \mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{s})$ we set
$A_{\mathbf{w}} = \{t \in \widetilde{EV}(\vec{s}) :\ (w_1, \dots, w_l, t) \notin \mathcal{F}\}$ and $A_\emptyset = \{t \in \widetilde{EV}(\vec{s}) : (t) \notin \mathcal{F}\}$.
We define

$$(\mathcal{F})^1_{\vec{s}} = \{\mathbf{w} \in \mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{s}) \cup \{\emptyset\} : A_{\mathbf{w}} \text{ does not contain an infinite orderly sequence}\}.$$

It is easy to verify that $(\mathcal{F})^1_{\vec{s}}$ is hereditary, hence it is pointwise closed since $\mathcal{F}$ is pointwise closed (Proposition 3.14). So, we can define for every $\xi > 1$ the ξ-derivatives of $\mathcal{F}$ recursively as follows:

$(\mathcal{F})^{\zeta+1}_{\vec{s}} = ((\mathcal{F})^{\zeta}_{\vec{s}})^1_{\vec{s}}$ for all $\zeta < \omega_1$, and
$(\mathcal{F})^{\xi}_{\vec{s}} = \bigcap_{\beta<\xi} (\mathcal{F})^{\beta}_{\vec{s}}$ for ξ a limit ordinal.

The *strong Cantor-Bendixson index* $sO_{\vec{s}}(\mathcal{F})$ of $\mathcal{F}$ on $\vec{s}$ is the smallest countable ordinal ξ such that $(\mathcal{F})^{\xi}_{\vec{s}} = \emptyset$.

Remark 3.16. Let $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and let $\mathcal{F}, \mathcal{R} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ be nonempty, hereditary and pointwise closed families.

(i) $sO_{\vec{s}}(\mathcal{F})$ is a countable successor ordinal less than or equal to the "usual" Cantor-Bendixson index $O(\mathcal{F})$ of $\mathcal{F}$ into $\{0,1\}^{\widetilde{L}(\Sigma,\vec{k};\upsilon)}$ (cf. [KM]).
(ii) $sO_{\vec{s}}(\mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{s})) = sO_{\vec{s}}(\mathcal{F})$.
(iii) $sO_{\vec{s}}(\mathcal{F}) \leq sO_{\vec{s}}(\mathcal{R})$ if $\mathcal{F} \subseteq \mathcal{R}$.
(iv) If $\vec{t} = (t_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ with $(t_{k+n})_{n\in\mathbb{N}} \prec \vec{s}$ for some $k \in \mathbb{N} \cup \{0\}$ and $\mathbf{w} \in (\mathcal{F})^{\xi}_{\vec{s}}$, then for every $\mathbf{w}_1 \in \widetilde{EV}^{<\infty}(\vec{t})$ such that $\mathbf{w}_1 \in \widetilde{EV}^{<\infty}(\mathbf{w})$ we have that $\mathbf{w}_1 \in (\mathcal{F})^{\xi}_{\vec{t}}$.
(v) If $\vec{s}_1 \prec \vec{s}$, then $sO_{\vec{s}_1}(\mathcal{F}) \geq sO_{\vec{s}}(\mathcal{F})$, according to (iv).
(vi) If $r(\vec{s}_1) \setminus r(\vec{s})$ is a finite set, then $sO_{\vec{s}_1}(\mathcal{F}) \geq sO_{\vec{s}}(\mathcal{F})$.

In Proposition 3.18 below, we will prove that the corresponding strong Cantor-Bendixson index to Schreier families of order ξ is equal to $\xi+1$. For the proof of Proposition 3.18 we will need the following lemma, which is analogous to Lemma 2.8 in [F5] and has an analogous proof.

Lemma 3.17. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $1 \leq \xi$ a countable ordinal, $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, $\vec{s}_1 \prec \vec{s}$ and let $\mathcal{F} \subseteq \widetilde{EV}^{<\infty}(\vec{s})$ such that $\mathcal{F}_*$ and $(\mathcal{F}(t))_*$ are pointwise closed for every $t \in \widetilde{EV}(\vec{s})$.*

(i) *$((\mathcal{F}(t))_*)^{\xi}_{\vec{s}_1} \subseteq (\mathcal{F}_*)^{\xi}_{\vec{s}_1}(t)$ for every $t \in \widetilde{EV}(\vec{s})$.*
(ii) *If $\mathbf{w} = (w_1, \dots, w_l) \neq \emptyset$ and $\mathbf{w} \in (\mathcal{F}_*)^{\xi}_{\vec{s}_1}$, then there exist $\vec{s}_2 \prec \vec{s}_1$ and $t \in \widetilde{EV}(\vec{s})$ with $t <_{R_1} w_1$ or $dom(t) \subseteq dom(w_1)$ such that $\mathbf{w} - t \in ((\mathcal{F}(t))_*)^{\xi}_{\vec{s}_2}$.*

Proof. (i) This can be proved by induction on ξ, using Definition 3.15 and that $(\mathcal{F}(t))_* \subseteq \mathcal{F}_*(t)$.

(ii) The proof is by induction on ξ. The main argument is contained in the proof of the case $\xi = 1$. Let $\mathbf{w} = (w_1, \dots, w_l) \in (\mathcal{F}_*)^1_{\vec{s}_1}$. For every $u \in \widetilde{EV}(\vec{s}_1) \setminus A_{\mathbf{w}}$ there exists $v_u = (v_1^u, \dots, v_{l_u}^u) \in \mathcal{F}$ such that $(w_1, \dots, w_l, u) \in \widetilde{EV}^{<\infty}(v_u)$. Then $v_1^u \leq w_1$ (which means that $v_1^u <_{R_1} w_1$ or $dom(v_1^u) \subseteq dom(w_1)$) for every $u \in \widetilde{EV}(\vec{s}_1) \setminus A_{\mathbf{w}}$ and the set $\{v \in \widetilde{EV}(\vec{s}) : \ v \leq w_1\}$ is finite. Since $\mathbf{w} \in (\mathcal{F}_*)^1_{\vec{s}_1}$, according to Theorem 2.6 (case $m = 1$), there exists $\vec{s}_2 \prec \vec{s}_1$ and $t \in \widetilde{EV}(\vec{s})$ with $t \leq w_1$ such that $\widetilde{EV}(\vec{s}_2) \subseteq \widetilde{EV}(\vec{s}_1) \setminus A_{\mathbf{w}}$ and $v_1^u = t$ for every $u \in \widetilde{EV}(\vec{s}_2)$. Then $\mathbf{w} - t \in ((\mathcal{F}(t))_*)^1_{\vec{s}_2}$. □

Proposition 3.18. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\xi < \omega_1$ be an ordinal and $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Then,*

$$sO_{\vec{s}_1}\Big((\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_*\Big) = \xi + 1 \textit{ for every } \vec{s}_1 \prec \vec{s}.$$

Proof. We have $(\widetilde{L}^0(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_* = \{\emptyset\}$ for every $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and $sO_{\vec{s}_1}(\{\emptyset\}) = 1$, for $\vec{s}_1 \prec \vec{s}$, since $(\{\emptyset\})^1_{\vec{s}} = \emptyset$, so, the conclusion holds for $\xi = 0$.

The families $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_*$, $(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s})(t))_*$ are hereditary and pointwise closed for every $1 \leq \xi \leq \omega_1$, $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and $t \in \widetilde{EV}(\vec{s})$, according to Proposition 3.14, since in case $\min dom^+(t) = n$, Proposition 3.4 implies that

$$(\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))(t) = \widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s} - t) \text{ for some } \xi_n < \xi.$$

In order to prove this proposition, it is enough to prove by induction on ξ that $\Big((\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^\xi_{\vec{s}_1} = \{\emptyset\}$ for every $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, $\vec{s}_1 \prec \vec{s}$ and $1 \leq \xi < \omega_1$. Since $(\widetilde{L}^1(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_* = \{(t) : \ t \in \widetilde{EV}(\vec{s})\} \cup \{\emptyset\}$, we have $\Big((\widetilde{L}^1(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^1_{\vec{s}_1} = \{\emptyset\}$ for every $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and $\vec{s}_1 \prec \vec{s}$.

Let $\xi > 1$ and assume that $\Big((\widetilde{L}^\zeta(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^\zeta_{\vec{s}_1} = \{\emptyset\}$ for every $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, $\vec{s}_1 \prec \vec{s}$ and $1 \leq \zeta < \xi$. Let $\vec{s} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ and $\vec{s}_1 \prec \vec{s}$. For every $t \in \widetilde{EV}(\vec{s})$ with $\min dom^+(t) = n$ we have, according to Remark 3.16 (vi), that

$$\Big((\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))(t)_*\Big)^{\xi_n}_{\vec{s}_1} = \Big((\widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s} - t))_*\Big)^{\xi_n}_{\vec{s}_1} = \Big((\widetilde{L}^{\xi_n}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s} - t))_*\Big)^{\xi_n}_{\vec{s}_1 - t} = \{\emptyset\}.$$

This gives that $\emptyset \in \Big((\widetilde{L}^\xi(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{s}))(t)_*\Big)^{\xi_n}_{\vec{s}_1}$. According to Lemma 3.17

(i) we have that $(t) \in \Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\xi_n}_{\vec{s}_1}$. Hence, $\emptyset \in \Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap \widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. Indeed, if $\xi=\zeta+1$, then $(t)\in\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\zeta}_{\vec{s}_1}$ for every $t\in\widetilde{EV}(\vec{s}_1)$ and if ξ is a limit ordinal, then $\sup\xi_n=\xi$ and $\emptyset\in\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\xi_n}_{\vec{s}_1}$ for every $n\in\mathbb{N}$.

We will prove that $\{\emptyset\}=\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}$ for every $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. Indeed, let $\mathbf{w}=(w_1,\dots,w_l)\in\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}$ for some $l\in\mathbb{N}$, $\vec{s}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$ and $\vec{s}_1\prec\vec{s}$. According to Lemma 3.17 (ii) there exists $\vec{s}_2\prec\vec{s}_1$ and $t\in\widetilde{EV}(\vec{s})$ such that $\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))(t)_*\Big)^{\xi}_{\vec{s}_2}\neq\emptyset$. If $\min dom^+(t)=n$, analogously to the previous paragraph we have,

$$\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))(t)_*\Big)^{\xi}_{\vec{s}_2}=\Big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}-t))_*\Big)^{\xi}_{\vec{s}_2}=\Big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}-t))_*\Big)^{\xi}_{\vec{s}_2-t}\neq\emptyset.$$

A contradiction, since $\xi_n<\xi$ and $\Big((\widetilde{L}^{\xi_n}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}-t))_*\Big)^{\xi}_{\vec{s}_2-t}=\emptyset$, according to the induction hypothesis.

Hence, $\Big((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*\Big)^{\xi}_{\vec{s}_1}=\{\emptyset\}$ and $sO_{\vec{s}_1}((\widetilde{L}^{\xi}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{s}))_*)=\xi+1$ for every $\xi<\omega_1$. □

Corollary 3.19. *Let ξ_1,ξ_2 be countable ordinals with $\xi_1<\xi_2$ and $\vec{w}\in\widetilde{L}^{\infty}(\Sigma,\vec{k};\upsilon)$. Then there exists $\vec{u}\prec\vec{w}$ such that*

$$(\widetilde{L}^{\xi_1}(\Sigma,\vec{k};\upsilon))_*\cap\widetilde{EV}^{<\infty}(\vec{u})\subseteq(\widetilde{L}^{\xi_2}(\Sigma,\vec{k};\upsilon))^*\setminus\widetilde{L}^{\xi_2}(\Sigma,\vec{k};\upsilon).$$

Proof. The family $(\widetilde{L}^{\xi_1}(\Sigma,\vec{k};\upsilon))_*\subseteq\widetilde{L}^{<\infty}(\Sigma,\vec{k};\upsilon)$ is a tree. According to Theorem 3.5 and Proposition 3.12 there exists $\vec{u}\prec\vec{w}$ such that:

either $\widetilde{L}^{\xi_2}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{u})\subseteq(\widetilde{L}^{\xi_1}(\Sigma,\vec{k};\upsilon))_*$,

or $(\widetilde{L}^{\xi_1}(\Sigma,\vec{k};\upsilon))_*\cap\widetilde{EV}^{<\infty}(\vec{u})\subseteq(\widetilde{L}^{\xi_2}(\Sigma,\vec{k};\upsilon))^*\setminus\widetilde{L}^{\xi_2}(\Sigma,\vec{k};\upsilon)$.

The first alternative of the dichotomy is impossible, since, according to Proposition 3.18

$$\xi_2+1=sO_{\vec{u}}((\widetilde{L}^{\xi_2}(\Sigma,\vec{k};\upsilon)\cap\widetilde{EV}^{<\infty}(\vec{u}))_*)\leq sO_{\vec{u}}((\widetilde{L}^{\xi_1}(\Sigma,\vec{k};\upsilon))_*)=\xi_1+1. \quad \square$$

The following Theorem 3.21, the main result in this Section, refines Theorem 3.5 in case the partition family is a tree. We note by $\sup X$, where X is a set of ordinals, the least ordinal α such that $\beta\leq\alpha$ for every $\beta\in X$.

Definition 3.20. Let Σ, υ and $\vec{k}$ satisfy the standard assumptions and $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$. We set $\mathcal{F}_h = \{\mathbf{w} \in \mathcal{F} : \widetilde{EV}^{<\infty}(\mathbf{w}) \subseteq \mathcal{F}\} \cup \{\emptyset\}$.

Of course, $\mathcal{F}_h$ is the largest subfamily of $\mathcal{F} \cup \{\emptyset\}$ which is hereditary.

Theorem 3.21. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions, $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ a family which is a tree and $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. Then we have the following cases:*

[Case 1] *The family $\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{w})$ is not pointwise closed.*

Then, there exists $\vec{u} \prec \vec{w}$ such that $\widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

[Case 2] *The family $\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{w})$ is pointwise closed.*

Then, setting

$$\zeta^{\mathcal{F}}_{\vec{w}} = \sup\{sO_{\vec{u}}(\mathcal{F}_h) : \vec{u} \prec \vec{w}\},$$

which is a countable ordinal, the following subcases obtain:

2(i) *If $\xi + 1 < \zeta^{\mathcal{F}}_{\vec{w}}$, then there exists $\vec{u} \prec \vec{w}$ such that*

$$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F};$$

2(ii) *if $\xi + 1 > \xi > \zeta^{\mathcal{F}}_{\vec{w}}$, then for every $\vec{w}_1 \prec \vec{w}$ there exists $\vec{u} \prec \vec{w}_1$ such that*
$\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$;
(equivalently $\mathcal{F} \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$) and

2(iii) *if $\xi + 1 = \zeta^{\mathcal{F}}_{\vec{w}}$ or $\xi = \zeta^{\mathcal{F}}_{\vec{w}}$, then there exists $\vec{u} \prec \vec{w}$ such that either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.*

Proof. **[Case 1]** If the hereditary family $\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{w})$ is not pointwise closed, then, according to Proposition 3.14, there exists $\vec{u} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ such that $\widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{w}) \subseteq \mathcal{F}$. Of course, $\vec{u} \prec \vec{w}$.

[Case 2] If the hereditary family $\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{w})$ is pointwise closed, then $\zeta^{\mathcal{F}}_{\vec{w}}$ is a countable ordinal, since the "usual" Cantor-Bendixson index $O(\mathcal{F}_h)$ of $\mathcal{F}_h$ into $\{0,1\}^{\widetilde{L}(\Sigma, \vec{k}, \upsilon}$ is countable (Remark 3.16(i)) and also $sO_{\vec{u}}(\mathcal{F}_h) \leq O(\mathcal{F}_h)$ for every $\vec{u} \prec \vec{w}$.

2(i) Let $\xi + 1 < \zeta^{\mathcal{F}}_{\vec{w}}$. Then there exists $\vec{u}_1 \prec \vec{w}$ such that $\xi + 1 < sO_{\vec{u}_1}(\mathcal{F}_h)$. According to Theorem 3.5 and Proposition 3.12, there exists $\vec{u} \prec \vec{u}_1$ such that

either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}_h \subseteq \mathcal{F}$,

or $\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \setminus \widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))_*$.

The second alternative is impossible. Indeed, if $\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))_*$, then, according to Remark 3.16 and Proposition 3.18,

$sO_{\vec{u}_1}(\mathcal{F}_h) \leq sO_{\vec{u}}(\mathcal{F}_h) = sO_{\vec{u}}(\mathcal{F}_h \cap \widetilde{EV}^{<\infty}(\vec{u})) \leq sO_{\vec{u}}((\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))_*) = \xi + 1$;

a contradiction. Hence, $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

2(ii) Let $\xi + 1 > \xi > \zeta^{\mathcal{F}}_{\vec{w}}$ and $\vec{w}_1 \prec \vec{w}$. According to Theorem 3.5, there exists $\vec{u_1} \prec \vec{w}_1$ such that

either $\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}_1) \subseteq \mathcal{F}_h$,

or $\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u_1}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_h$.

Proposition 3.18 gives $\zeta^{\mathcal{F}}_{\vec{w}} + 1 = sO_{\vec{u}_1}((\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}_1))_*)$. Since $sO_{\vec{u}_1}(\mathcal{F}_h) \leq \zeta^{\mathcal{F}}_{\vec{w}}$, the first alternative is impossible, according to Remark 3.16 (*iii*). So,

(1) $$\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}_1) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_h .$$

According to Theorem 3.5, there exists $\vec{u} \prec \vec{u}_1$ such that

either $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,

or $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.

Since the family $\mathcal{F}$ is a tree, the first alternative does not hold. Indeed, if $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$, then $(\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}))^* \subseteq \mathcal{F}^* = \mathcal{F}$. Consequently, from Proposition 3.11 follows that $(\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \cap \widetilde{EV}^{<\infty}(\vec{u}) = (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon) \cap \widetilde{EV}^{<\infty}(\vec{u}))^* \subseteq \mathcal{F}$. Since $\xi > \zeta^{\mathcal{F}}_{\vec{w}}$, according to Corollary 3.19, there exists $\vec{t} \prec \vec{u}$ such that

$(\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon))_* \cap \widetilde{EV}^{<\infty}(\vec{t}) \subseteq (\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon))^* \cap \widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$.

Hence, $(\widetilde{L}^{\zeta^{\mathcal{F}}_{\vec{w}}}(\Sigma, \vec{k}; \upsilon))_* \cap \widetilde{EV}^{<\infty}(\vec{t}) \subseteq \mathcal{F}_h$. This is a contradiction to the relation (1).

2(iii) In the cases $\zeta^{\mathcal{F}}_{\vec{w}} = \xi + 1$ or $\zeta^{\mathcal{F}}_{\vec{w}} = \xi$, we use Theorem 3.5. □

The following immediate corollary to Theorem 3.21 is more useful for applications. A quite simplified consequence of Theorem 3.21, one not involving Schreier-type families of variable ω-$\mathbb{Z}^*$-located words is equivalent to the particular case of Theorem 2.5 in which the partition families of $\widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$ are clopen sets in the product topology (Theorem 3.23 below).

Corollary 3.22. *Let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ which is a tree and let $\vec{w} \in \widetilde{L}^{\infty}(\Sigma, \vec{k}; \upsilon)$. Then*

(i) *either there exists $\vec{u} \prec \vec{w}$ such that $\widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,*
(ii) *or for every countable ordinal $\xi > \zeta^{\mathcal{F}}_{\vec{w}}$ there exists $\vec{u} \prec \vec{w}$, such that for every $\vec{u}_1 \prec \vec{u}$ the unique initial segment of $\vec{u}_1$ which is an element of $\widetilde{L}^{\xi}(\Sigma, \vec{k}; \upsilon)$ belongs to $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.*

Now, we will prove that Corollary 3.22 can be considered as a strengthening of Theorem 3.23, giving a reformulation of Theorem 3.23 in Theorem 3.24 below.

Theorem 3.23. *Let Σ, υ and $\vec{k}$ satisfy the standard assumptions. If $\mathcal{U} \subseteq \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ is a pointwise closed family and $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$, then there exists $\vec{u} \prec \vec{w}$ such that*

$$\text{either } \widetilde{EV}^\infty(\vec{u}) \subseteq \mathcal{U}, \quad \text{or} \quad \widetilde{EV}^\infty(\vec{u}) \subseteq \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{U}.$$

Theorem 3.23 has the following reformulation.

Theorem 3.24. *Let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ which is a tree and let $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Then*

(i) *either there exists $\vec{u} \prec \vec{w}$ such that $\widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,*
(ii) *or there exists $\vec{u} \prec \vec{w}$ such that for every $\vec{u}_1 \prec \vec{u}$ there exists an initial segment of $\vec{u}_1$ which belongs to $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.*

Remark 3.25. (1) Theorem 3.24 implies Theorem 3.23. Indeed, let $\mathcal{U} \subseteq \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$ be a closed family in the product topology and $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Set

$$\mathcal{F}_{\mathcal{U}} = \{\mathbf{w} \in \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \colon \text{there exist } \vec{s} \in \mathcal{U} \text{ such that } \mathbf{w} \propto \vec{s}\}.$$

Since the family $\mathcal{F}_{\mathcal{U}}$ is a tree, we use Theorem 3.22. So, we have the following two cases:

[Case 1] There exists $\vec{u} \prec \vec{w}$ such that $\widetilde{EV}^{<\infty}(\vec{u}) \subseteq \mathcal{F}_{\mathcal{U}}$. Then, $\widetilde{EV}^\infty(\vec{u}) \subseteq \mathcal{U}$. Indeed, if $\vec{z} = (z_n)_{n\in\mathbb{N}} \in \widetilde{EV}^\infty(\vec{u})$, then $(z_1, \dots, z_n) \in \mathcal{F}_{\mathcal{U}}$ for every $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$ there exists $\vec{s}_n \in \mathcal{U}$ such that $(z_1, \dots, z_n) \propto \vec{s}_n$. Since $\mathcal{U}$ is pointwise closed, we have that $\vec{z} \in \mathcal{U}$ and consequently that $\widetilde{EV}^\infty(\vec{u}) \subseteq \mathcal{U}$.

[Case 2] There exists $\vec{u} \prec \vec{w}$ such that for every $\vec{u}_1 \prec \vec{u}$ there exists an initial segment of $\vec{u}_1$ which belongs to $\widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}_{\mathcal{U}}$. Hence, $\widetilde{EV}^\infty(\vec{u}) \subseteq \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{U}$.

(2) Theorem 3.23 implies Theorem 3.24. Indeed, let $\mathcal{F} \subseteq \widetilde{L}^{<\infty}(\Sigma, \vec{k}; \upsilon)$ which is a tree and let $\vec{w} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon)$. Set

$$\mathcal{U}_{\mathcal{F}} = \{\vec{t} = (t_n)_{n\in\mathbb{N}} \in \widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) : \text{there exist } k \in \mathbb{N} \text{ such that } (t_1, \dots, t_k) \in \mathcal{F}\}.$$

Since $\widetilde{L}^\infty(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{U}_{\mathcal{F}}$ is a closed family in the product topology, using Theorem 3.23, we obtain the conclusion of Theorem 3.24.

Using Corollary 3.22, we can get the respective result for variable ω-located words, which extends Corollary 3.8 and implies the particular case of Theorem 2.3 in case the partition family $\mathcal{F}$ is clopen.

Corollary 3.26. *Let $\mathcal{F} \subseteq L^{<\infty}(\Sigma, \vec{k}; \upsilon)$ which is a tree (see Definition 3.9 (iii)) and let $\vec{w} \in L^\infty(\Sigma, \vec{k}; \upsilon)$. Then*

(i) *either there exists* $\vec{u} \prec \vec{w}$ *such that* $EV^{<\infty}(\vec{u}) \subseteq \mathcal{F}$,

(ii) *or there exists* $\xi_0 < \omega_1$ *such that for every countable ordinal* $\xi > \xi_0$ *there exists* $\vec{u} \prec \vec{w}$, *such that for every* $\vec{u}_1 \prec \vec{u}$ *the unique initial segment of* $\vec{u}_1$ *which is an element of* $L^{\xi}(\Sigma, \vec{k}; \upsilon)$ *belongs to* $L^{<\infty}(\Sigma, \vec{k}; \upsilon) \setminus \mathcal{F}$.

4. Applications to the Ramsey theory of the rationals with addition

T. Budak, N. Işik and J. Pym in [BIP] (Theorem 4.2) introduced a representation of rational numbers with specific properties, according to which a non-zero rational number can be identified with a ω-$\mathbb{Z}^*$-located word over the alphabet $\Sigma = \{\alpha_n : \ n \in \mathbb{Z}^*\}$, where $\alpha_{-n} = \alpha_n = n$ for $n \in \mathbb{N}$, dominated by $(k_n)_{n\in\mathbb{Z}^*}$, where $k_{-n} = k_n = n$ for $n \in \mathbb{N}$. Hence, all the results concerning ω-$\mathbb{Z}^*$-located words over an alphabet $\Sigma = \{\alpha_n : \ n \in \mathbb{Z}^*\}$ dominated by a $(k_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}$, proved in the previous sections, can be formulating to statements concerning rational numbers.

In this section we present a strengthened van der Waerden theorem for the set of rational numbers, in Theorem 4.1, using Theorem 2.6 (case $m = 1$), an extended to every countable order Ramsey-type partition theorem for the set of rational numbers in Theorem 4.2 as a consequence of Theorem 3.5, and a partition theorem for infinite orderly sequences of rational numbers in Theorem 4.3 as a consequence of Theorem 2.5.

Analytically, according to [BIP], every rational number q has a unique expression in the form

$$\sum_{s=1}^{\infty} q_{-s}\frac{(-1)^s}{(s+1)!} \ + \ \sum_{r=1}^{\infty} q_r(-1)^{r+1}r!$$

where $(q_n)_{n\in\mathbb{Z}^*} \subseteq \mathbb{N}\cup\{0\}$ with $0 \leq q_{-s} \leq s$ for every $s > 0$, $0 \leq q_r \leq r$ for every $r > 0$ and $q_{-s} = q_r = 0$ for all but finite many r, s. So, for a non-zero rational number q, there exist unique $l \in \mathbb{N}$, $\{t_1 < \ldots < t_l\} = dom(q) \in [\mathbb{Z}^*]^{<\omega}_{>0}$ and $\{q_{t_1}, \ldots, q_{t_l}\} \subseteq \mathbb{N}$ with $1 \leq q_{t_i} \leq -t_i$ if $t_i < 0$ and $1 \leq q_{t_i} \leq t_i$ if $t_i > 0$ for every $1 \leq i \leq l$, such that defining $dom^-(q) = \{t \in dom(q) : \ t < 0\}$ and $dom^+(q) = \{t \in dom(q) : \ t > 0\}$ to have

$$q = \sum_{t\in dom^-(q)} q_t\frac{(-1)^{-t}}{(-t+1)!} \ + \sum_{t\in dom^+(q)} q_t(-1)^{t+1}t! \quad (\text{we set } \sum_{t\in\emptyset} = 0).$$

Observe that

$$e^{-1} - 1 = -\sum_{t=1}^{\infty}\frac{2t-1}{(2t)!} < \sum_{t\in dom^-(q)} q_t\frac{(-1)^{-t}}{(-t+1)!} < \sum_{t=1}^{\infty}\frac{2t}{(2t+1)!} = e^{-1}$$

and that

$$\sum_{t\in dom^+(q)} q_t(-1)^t(t+1)! \in \mathbb{Z}^* \ \text{ if } \ dom^+(q)\neq\emptyset.$$

Let $\alpha_{-n}=\alpha_n=n$ and $k_{-n}=k_n=n$ for $n\in\mathbb{N}$. We set $\Sigma=\{\alpha_n:\ n\in\mathbb{Z}^*\}$ and $\vec{k}=(k_n)_{n\in\mathbb{Z}^*}$. For $\upsilon=0$ we define the function $g:\widetilde{L}(\Sigma\cup\{0\},\vec{k})\to\mathbb{Q}$, which sends a word $w=q_{t_1}\dots q_{t_l}\in\widetilde{L}(\Sigma\cup\{0\},\vec{k})$ to the rational number

$$g(w)=\textstyle\sum_{t\in dom^-(w)} q_t\frac{(-1)^{-t}}{(-t+1)!}\ +\ \sum_{t\in dom^+(w)} q_t(-1)^{t+1}t!.$$

It is easy to see that the restriction of g to the set of the constant words $\widetilde{L}(\Sigma,\vec{k})$ is one-to-one and onto $\mathbb{Q}\setminus\{0\}$ and that $g(w_1\star w_2)=g(w_1)+g(w_2)$ for every $w_1<_{R_1} w_2\in\widetilde{L}(\Sigma\cup\{0\},\vec{k})$. Also, observe that, via the function g, each variable word $w=q_{t_1}\dots q_{t_l}\in\widetilde{L}(\Sigma,\vec{k};0)$ corresponds to a function q which sends every $(i,j)\in\mathbb{N}\times\mathbb{N}$ with $1\leq i\leq -\max dom^-(w)$, $1\leq j\leq \min dom^+(w)$, to

$$q(i,j)=g(T_{(j,i)}(w))=\sum_{t\in C^-} q_t\frac{(-1)^{-t}}{(-t+1)!}+i\sum_{t\in V^-}\frac{(-1)^{-t}}{(-t+1)!}+\sum_{t\in C^+} q_t(-1)^{t+1}t!+j\sum_{t\in V^+}(-1)^{t+1}t!,$$

where $C^-=\{t\in dom^-(w):\ q_t\in\Sigma\}$, $V^-=\{t\in dom^-(w):\ q_t=0\}$ and $C^+=\{t\in dom^+(w):\ q_t\in\Sigma\}$, $V^+=\{t\in dom^+(w):\ q_t=0\}$.

For two non-zero rational numbers $q_1,q_2\in g(\widetilde{L}(\Sigma,\vec{k}))$ we set

$$q_1\prec q_2 \iff g^{-1}(q_1)<_{R_1} g^{-1}(q_2).$$

Notation. Let $(X,+)$ arbitrary semigroup. For $(x_n)_{n\in\mathbb{N}}\subseteq X$ we set

$$FS[(x_n)_{n\in\mathbb{N}}]=\{x_{n_1}\ +\ \dots\ +\ x_{n_l}:\ n_1<\dots<n_l\in\mathbb{N}\}.$$

Theorem 4.1. *Let $\mathbb{Q}=Q_1\cup\ldots\cup Q_r$ for $r\in\mathbb{N}$. Then, there exist $1\leq i_0\leq r$ and for every $n\in\mathbb{N}$ a function $q_n:\{1,\dots,n\}\times\{1,\dots,n\}\cup\{(0,0)\}\to\mathbb{Q}$ with*

$$q_n(i,j)=\sum_{t\in C_n^-} q_t^n\frac{(-1)^{-t}}{(-t+1)!}+i\sum_{t\in V_n^-}\frac{(-1)^{-t}}{(-t+1)!}+\sum_{t\in C_n^+} q_t^n(-1)^{t+1}t!+j\sum_{t\in V_n^+}(-1)^{t+1}t!,$$

where $C_n^-,V_n^-\in[\mathbb{Z}^-]^{<\omega}_{>0}$, $C_n^+,V_n^+\in[\mathbb{N}]^{<\omega}_{>0}$ with $C_n^-\cap V_n^-=\emptyset=C_n^+\cap V_n^+$, $q_t^n\in\mathbb{N}$ with $1\leq q_t^n\leq -t$ for $t\in C_n^-$, $1\leq q_t^n\leq t$ for $t\in C_n^+$, which satisfy $q_n(i_n,j_n)\prec q_{n+1}(i_{n+1},j_{n+1})$ for every $n\in\mathbb{N}$, and

$$FS\big[\big(q_n(i_n,j_n)\big)_{n\in\mathbb{N}}\big]\subseteq Q_{i_0}$$

for all $((i_n,j_n))_{n\in\mathbb{N}}\subseteq\mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ with $0\leq i_n,j_n\leq n$ for every $n\in\mathbb{N}$.

Proof. Let the function $g:\widetilde{L}(\Sigma,\vec{k};0)\to\mathbb{Q}$ defined above. According to Theorem 2.6 there exists a sequence $(w_n)_{n\in\mathbb{N}}\in\widetilde{L}^\infty(\Sigma,\vec{k};0)$ and $1\leq i_0\leq r$, such that $T_{(i_1,j_1)}(w_{n_1})\star\ldots\star T_{(i_\lambda,j_\lambda)}(w_{n_\lambda})\in g^{-1}(Q_{i_0})$, for every $\lambda\in\mathbb{N}$, $n_1<\ldots<n_\lambda\in\mathbb{N}$, $(i_l,j_l)\in\mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ such that $0\leq i_l,j_l\leq n_l$, for

every $1 \leq l \leq \lambda$ and $(0,0) \in \{(i_1, j_1), \ldots, (i_\lambda, j_\lambda)\}$. Let $w_n = w^n_{m^n_1} \ldots w^n_{m^n_{l_n}}$ for every $n \in \mathbb{N}$. Set $q_n(i,j) = g(w_{3n-2} \star T_{(j,i)}(w_{3n-1}) \star T_{(1,1)}(w_{3n}))$ for every $n \in \mathbb{N}$ and $(i,j) \in \mathbb{N} \times \mathbb{N} \cup \{(0,0)\}$ with $0 \leq i, j \leq n$. The functions q_n satisfy the required properties. □

Notation. For an arbitrary semigroup $(X, +)$ and a sequence $(x_n)_{n\in\mathbb{N}} \subseteq X$, for $y_1 = x_{n_1} + \ldots + x_{n_l}$, $y_2 = x_{m_1} + \ldots + x_{m_\nu} \in FS[(x_n)_{n\in\mathbb{N}}]$ we write $y_1 < y_2$ if $n_l < m_1$, and
$\big[FS[(x_n)_{n\in\mathbb{N}}]\big]^{<\infty}_{>0} = \{(y_1, \ldots, y_m) : \ m \in \mathbb{N}, \ y_1 < \ldots < y_m \in FS[(x_n)_{n\in\mathbb{N}}]\}$.
For every countable ordinal $\xi \geq 1$ and every $n \in \mathbb{N}$ we set

$\mathbb{Q}^{<\infty} = \{(q_1, \ldots, q_l) : \ l \in \mathbb{N}, \ q_1 \prec \ldots \prec q_l \in \mathbb{Q} \setminus \{0\}\} \cup \{\emptyset\}$, and

$\mathbb{Q}^{\xi} = \{(q_1, \ldots, q_l) \in \mathbb{Q}^{<\infty} : \ \{\min dom^+(q_1), \ldots, \min dom^+(q_l)\} \in \mathcal{A}_\xi\}$.

Combining Theorem 3.5 with the representation of rational numbers via the function g, analogously to Theorem 4.1, we get the following Ramsey type partition theorem for every countable order ξ for the set of rational numbers. The case $\xi = 1$ corresponds to Theorem 4.1.

Theorem 4.2. *Let $\xi \geq 1$ be a countable ordinal and a family $G \subseteq \mathbb{Q}^{<\infty}$. Then, for each $n \in \mathbb{N}$ there exists a function $q_n : \{1, \ldots, n\} \times \{1, \ldots, n\} \cup \{(0,0)\} \to \mathbb{Q}$ with*

$$q_n(i,j) = \sum_{t\in C_n^-} q_t^n \frac{(-1)^{-t}}{(-t+1)!} + i \sum_{t\in V_n^-} \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t\in C_n^+} q_t^n (-1)^{t+1} t! + j \sum_{t\in V_n^+} (-1)^{t+1} t!,$$

where $C_n^-, V_n^- \in [\mathbb{Z}^-]^{\leq\omega}_{>0}$, $C_n^+, V_n^+ \in [\mathbb{N}]^{\leq\omega}_{>0}$ with $C_n^- \cap V_n^- = \emptyset = C_n^+ \cap V_n^+$, $q_t^n \in \mathbb{N}$ with $1 \leq q_t^n \leq -t$ for $t \in C_n^-$, $1 \leq q_t^n \leq t$ for $t \in C_n^+$, which satisfy $q_n(i_n, j_n) \prec q_{n+1}(i_{n+1}, j_{n+1})$ for every $n \in \mathbb{N}$, and

$$\text{either } \mathbb{Q}^{\xi} \cap \big[FS\big[\big(q_n(i_n, j_n)\big)_{n\in\mathbb{N}}\big]\big]^{<\infty}_{>0} \subseteq G,$$
$$\text{or } \mathbb{Q}^{\xi} \cap \big[FS\big[\big(q_n(i_n, j_n)\big)_{n\in\mathbb{N}}\big]\big]^{<\infty}_{>0} \subseteq \mathbb{Q}^{<\infty} \setminus G$$

for all $((i_n, j_n))_{n\in\mathbb{N}} \subseteq \mathbb{N} \times \mathbb{N} \cup \{(0,0)\}$ with $0 \leq i_n, j_n \leq n$ for every $n \in \mathbb{N}$.

Notation. For a semigroup $(X, +)$ and $(x_n)_{n\in\mathbb{N}} \subseteq X$, we set
$\big[FS[(x_n)_{n\in\mathbb{N}}]\big]^{\mathbb{N}} = \{(y_n)_{n\in\mathbb{N}} : \ y_n \in FS[(x_n)_{n\in\mathbb{N}}]$ and $y_n < y_{n+1}$ for every $n \in \mathbb{N}\}$.

As a corollary of Theorem 2.5 we have the following partition theorem for infinite ordered sequences of rational numbers.

Theorem 4.3. *Let $\mathcal{U}$ be a Borel subset of $\mathbb{Q}^{\mathbb{N}}$ (in the product topology considering $\mathbb{Q}$ with the discrete topology). Then, for each $n \in \mathbb{N}$ there exists* *a function* $q_n : \{1,\dots,n\}\times\{1,\dots,n\}\cup\{(0,0)\}\to\mathbb{Q}$ *with*

$$q_n(i,j)=\sum_{t\in C_n^-} q_t^n\frac{(-1)^{-t}}{(-t+1)!}+i\sum_{t\in V_n^-}\frac{(-1)^{-t}}{(-t+1)!}+\sum_{t\in C_n^+} q_t^n(-1)^{t+1}t!+j\sum_{t\in V_n^+}(-1)^{t+1}t!,$$

where $C_n^-,V_n^-\in[\mathbb{Z}^-]^{<\omega}_{>0}$, $C_n^+,V_n^+\in[\mathbb{N}]^{<\omega}_{>0}$ *with* $C_n^-\cap V_n^-=\emptyset=C_n^+\cap V_n^+$, $q_t^n\in\mathbb{N}$, *with* $1\le q_t^n\le -t$ *for* $t\in C_n^-$, $1\le q_t^n\le t$ *for* $t\in C_n^+$, *which satisfy* $q_n(i_n,j_n)\prec q_{n+1}(i_{n+1},j_{n+1})$ *for every* $n\in\mathbb{N}$, *and*

$$\text{either } \big[FS[(q_n(i_n,j_n))_{n\in\mathbb{N}}]\big]^{\mathbb{N}}\subseteq\mathcal{U}, \ \text{ or } \big[FS[(q_n(i_n,j_n))_{n\in\mathbb{N}}]\big]^{\mathbb{N}}\subseteq\mathbb{Q}^{\mathbb{N}}\setminus\mathcal{U}$$

for all $(i_n,j_n)_{n\in\mathbb{N}}\subseteq\mathbb{N}\times\mathbb{N}\cup\{(0,0)\}$ *with* $0\le i_n,j_n\le n$ *for every* $n\in\mathbb{N}$.

Proof. Let the alphabet $\Sigma=\{\alpha_n : n\in\mathbb{Z}^*\}$, $\vec{k}=(k_n)_{n\in\mathbb{Z}^*}$, where $\alpha_{-n}=\alpha_n=n$ and $k_{-n}=k_n=n$ for every $n\in\mathbb{N}$, and $\upsilon=0$. We set $\hat{g}:\widetilde{L}^\infty(\Sigma,\vec{k};0)\to\mathbb{Q}^{\mathbb{N}}$ with $\hat{g}((w_n)_{n\in\mathbb{N}})=(g(w_n))_{n\in\mathbb{N}}$. The family $\hat{g}^{-1}(\mathcal{U})\subseteq\widetilde{L}^\infty(\Sigma,\vec{k};0)$ is pointwise closed, since the function $\hat{g}$ is continuous. So, according to Theorem 2.5, there exists $\vec{w}=(w_n)_{n\in\mathbb{N}}\in\widetilde{L}^\infty(\Sigma,\vec{k};0)$ such that

$$\text{either } \widetilde{EV}^{\infty}(\vec{w})\subseteq\hat{g}^{-1}(\mathcal{U}), \text{ or } \widetilde{EV}^{\infty}(\vec{w})\subseteq\widetilde{L}^\infty(\Sigma,\vec{k};0)\setminus\hat{g}^{-1}(\mathcal{U}).$$

From this point on, the proof is analogous to the one of Theorem 4.1. □

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