

DENSITY THEOREMS FOR RATIONAL NUMBERS

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ABSTRACT. We introduce the notion of a rational system of measure preserving transformations and we prove a recurrence result for such systems. Using this result, and exploiting the theory of semigroups developed in [HS], we give sufficient conditions in order a subset of rational numbers to contain arbitrary long arithmetic progressions. Also, using the theory of words over a finite alphabet, we prove that dense subsets of rational numbers contain specific arithmetic configurations.

INTRODUCTION

In 1927, van der Waerden proved (in [vdW]) that for any finite partition of the set of natural numbers, there exists a cell of the partition which contains arbitrary long arithmetic progressions, which is a (perhaps the most) fundamental result of Ramsey theory. The density version of the van der Waerden theorem, that any set of positive upper density in \mathbb{N} possesses arbitrary long arithmetic progressions (the *upper density* of a subset $A \subseteq \mathbb{N}$ is defined by $\bar{d}(A) = \limsup_n \frac{|A \cap \{1, \dots, n\}|}{n}$, where $|A|$ denotes the cardinality of A) was conjectured by Erdős and Turán in 1930's and established by Szemerédi in 1975 ([Sz]).

Furstenberg, in 1977, reproved Szemerédi's theorem, introducing a correspondence principle, which provides the link between density Ramsey theory and ergodic theory ([Fu]) and proving a multiple recurrence theorem for measure preserving systems.

In this paper we will prove a multiple recurrence and two density results for the semigroup of rational numbers giving sufficient conditions in order a set of rational numbers to contain arbitrary long arithmetic progressions. Using a representation of rational numbers (proved in [BIP]), according to which every rational number can be represented as a dominated located word over an infinite alphabet, we define the notion of a rational system (Definition 1.3). We obtain:

(a) a multiple recurrence result concerning rational systems of measure preserving transformations in Theorem 1.4, using the analogous result of Furstenberg-Katznelson for the IP-systems of measure preserving transformations,

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(b) a sufficient condition via Følner sequences in order a subset of rational numbers to contain arbitrary long arithmetic progressions in Theorem 2.4, using a result of Hindman and Strauss for infinite countable left cancellative semigroups; and

(c) a density result viewing the rational numbers as located words in Theorem 2.8, which follows from the density Hales-Jewett theorem of Furstenberg-Katznelson.

Notation. Let $\mathbb{N} = \{1, 2, \dots\}$ be the set of natural numbers, $\mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$ be the set of integer numbers, $\mathbb{Q} = \{\frac{m}{n} : m \in \mathbb{Z}, n \in \mathbb{N}\}$ the set of rational numbers and $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$, $\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$.

1. A MULTIPLE RECURRENCE RESULT FOR RATIONAL NUMBERS

In this section, using the representation of rational numbers as dominated located words over an infinite alphabet, we define (in Definition 1.3) the rational systems and we prove, in Theorem 1.4 below, a recurrence result for such systems for measure preserving transformations using an analogous result for IP-systems of Furstenberg-Katznelson.

According to [BIP], every rational number q has a unique expression in the form

$$q = \sum_{s=1}^{\infty} q_{-s} \frac{(-1)^s}{(s+1)!} + \sum_{r=1}^{\infty} q_r (-1)^{r+1} r!$$

where $(q_n)_{n \in \mathbb{Z}^*} \subseteq \mathbb{N} \cup \{0\}$ with $0 \leq q_{-s} \leq s$ for every $s > 0$, $0 \leq q_r \leq r$ for every $r > 0$ and $q_{-s} = q_r = 0$ for all but finite many r, s .

So, for a non-zero rational number q , there exist unique $l \in \mathbb{N}$, a non-empty finite subset of non-zero integers, the *domain* of q , $\{t_1 < \dots < t_l\} = \text{dom}(q) \in [\mathbb{Z}^*]_{>0}^{\leq \omega}$ and a subset of natural numbers $\{q_{t_1}, \dots, q_{t_l}\} \subseteq \mathbb{N}$ with $1 \leq q_{t_i} \leq -t_i$ if $t_i < 0$ and $1 \leq q_{t_i} \leq t_i$ if $t_i > 0$ for every $1 \leq i \leq l$, such that if $\text{dom}^-(q) = \{t \in \text{dom}(q) : t < 0\}$ and $\text{dom}^+(q) = \{t \in \text{dom}(q) : t > 0\}$ to have

$$q = \sum_{t \in \text{dom}^-(q)} q_t \frac{(-1)^{-t}}{(-t+1)!} + \sum_{t \in \text{dom}^+(q)} q_t (-1)^{t+1} t! \quad (\text{we set } \sum_{t \in \emptyset} = 0).$$

Consequently, q can be represented as the word

$$q = q_{t_1} \dots q_{t_l}.$$

It is easy to see that

$$e^{-1} - 1 = - \sum_{t=1}^{\infty} \frac{2t-1}{(2t)!} < \sum_{t \in \text{dom}^-(q)} q_t \frac{(-1)^{-t}}{(-t+1)!} < \sum_{t=1}^{\infty} \frac{2t}{(2t+1)!} = e^{-1}$$

and that

$$\sum_{t \in \text{dom}^+(q)} q_t (-1)^t (t+1)! \in \mathbb{Z}^* \text{ if } \text{dom}^+(q) \neq \emptyset.$$

We will now recall the notion of IP-systems introduced by Furstenberg and Katznelson in [FuKa].

Definition 1.1. Let $\{T_n\}_{n \in \mathbb{N}}$ be a set of commuting transformations of a space. To every multi-index $\alpha = \{t_1, \dots, t_l\} \in [\mathbb{N}]_{>0}^{\leq \omega}$, $t_1 < \dots < t_l$, we attach the transformation

$$\mathcal{T}_\alpha = T_{t_1} \dots T_{t_l}.$$

The corresponding family $\{\mathcal{T}_\alpha\}_{\alpha \in [\mathbb{N}]_{>0}^{\leq \omega}}$ is an *IP-system* (of transformations).

Two IP-systems $\{\mathcal{T}_\alpha^{(1)}\}_{\alpha \in [\mathbb{N}]_{>0}^{\leq \omega}}$, $\{\mathcal{T}_\alpha^{(2)}\}_{\alpha \in [\mathbb{N}]_{>0}^{\leq \omega}}$ defined by $\{T_n^{(1)}\}_{n \in \mathbb{N}}$ and $\{T_n^{(2)}\}_{n \in \mathbb{N}}$ respectively, are *commuting* if $T_n^{(1)} T_m^{(2)} = T_m^{(2)} T_n^{(1)}$ for every $n, m \in \mathbb{N}$.

Theorem 1.2 (Furstenberg-Katznelson, [FuKa]). *Let $\{\mathcal{T}_\alpha^{(1)}\}_{\alpha \in [\mathbb{N}]_{>0}^{\leq \omega}}, \dots, \{\mathcal{T}_\alpha^{(k)}\}_{\alpha \in [\mathbb{N}]_{>0}^{\leq \omega}}$ be k commuting IP-systems defined by the measure preserving transformations $\{T_n^{(j)}\}_{n \in \mathbb{N}}$, $1 \leq j \leq k$ of a measure space (X, \mathcal{B}, μ) with $\mu(X) = 1$ (i.e. $T_n^{(j)}$ is \mathcal{B} -measurable with $\mu(T_n^{(j)-1}(A)) = \mu(A)$ for every $A \in \mathcal{B}$, $1 \leq j \leq k$, $n \in \mathbb{N}$). If $A \in \mathcal{B}$ with $\mu(A) > 0$, then there exists an index α with*

$$\mu(A \cap \mathcal{T}_\alpha^{(1)-1}(A) \cap \dots \cap \mathcal{T}_\alpha^{(k)-1}(A)) > 0.$$

We will define the notion of a rational system (see also [K]), extending the notion of an IP-system.

Definition 1.3. Let $\{T_n\}_{n \in \mathbb{Z}^*}$ be a set of commuting transformations of a set X . For a non-zero rational number q represented as the word $q = q_{t_1} \dots q_{t_l}$, we define

$$\mathcal{T}^q(x) = T_{t_1}^{q_{t_1}} \dots T_{t_l}^{q_{t_l}}(x) \text{ and } \mathcal{T}^0(x) = x \text{ for every } x \in X.$$

The corresponding family $\{\mathcal{T}^q\}_{q \in \mathbb{Q}}$ is a *rational system* (of transformations).

Two rational systems $\{\mathcal{T}_1^q\}_{q \in \mathbb{Q}}$, $\{\mathcal{T}_2^q\}_{q \in \mathbb{Q}}$ defined by $\{T_{n,1}\}_{n \in \mathbb{Z}^*}$ and $\{T_{n,2}\}_{n \in \mathbb{Z}^*}$ respectively, are *commuting* if $T_{n,1} T_{m,2} = T_{m,2} T_{n,1}$ for every $n, m \in \mathbb{Z}^*$.

Using Theorem 1.2, we can take the following:

Theorem 1.4. *Let $\{\mathcal{T}_1^q\}_{q \in \mathbb{Q}}, \dots, \{\mathcal{T}_k^q\}_{q \in \mathbb{Q}}$ be k commuting rational systems defined by the measure preserving transformations $\{T_{n,j}\}_{n \in \mathbb{Z}^*}$, $1 \leq j \leq k$ respectively of a measure space (X, \mathcal{B}, μ) with $\mu(X) = 1$. If $A \in \mathcal{B}$ and $\mu(A) > 0$ then there exists $q \in \mathbb{Q}^*$ with*

$$\mu(A \cap (\mathcal{T}_1^q)^{-1}(A) \cap \dots \cap (\mathcal{T}_k^q)^{-1}(A)) > 0.$$

Proof. Let $A \in \mathcal{B}$ with $\mu(A) > 0$. For every $t \in \mathbb{Z}^*$ choose a natural number q_t with $1 \leq q_t \leq |t|$. For every $t \in \mathbb{N}$, $1 \leq i \leq k$, set $\phi_{2t-1}^{(i)} = T_{t,i}^{q_t}$, $\phi_{2t}^{(i)} = T_{-t,i}^{q-t}$ and let the corresponding IP-systems $\{\phi_\alpha^{(i)}\}_{\alpha \in [\mathbb{N}]_{>0}^{\leq \omega}}$.

According to Theorem 1.2 there exists $\alpha = \{t_1 < \dots < t_l\} \in [\mathbb{N}]_{>0}^{\leq \omega}$ with

$$\mu(A \cap \phi_\alpha^{(1)-1}(A) \cap \dots \cap \phi_\alpha^{(k)-1}(A)) > 0.$$

Since $\phi_\alpha^{(i)} = \mathcal{T}_i^q$ for all $1 \leq i \leq k$, where

$$q = \sum_{2t \in \alpha} q_t \frac{(-1)^{-t}}{(-t+1)!} + \sum_{2t-1 \in \alpha} q_t (-1)^{t+1} t! \in \mathbb{Q}^*,$$

we have that $\mu(A \cap (\mathcal{T}_1^q)^{-1}(A) \cap \dots \cap (\mathcal{T}_k^q)^{-1}(A)) > 0$. \square

Remark 1.5. With the same arguments as in Theorem 1.4, we can prove that:

- (1) There exists $q \in \mathbb{Z}^*$ which satisfies the conclusion of Theorem 1.4, setting $\phi_t^{(i)} = T_{t,i}^{q_t}$ for every $t \in \mathbb{N}$ and $1 \leq i \leq k$.
- (2) There exists $q \in (e^{-1} - 1, e^{-1}) \cap \mathbb{Q}^*$ which satisfies the conclusion of Theorem 1.4, setting $\phi_t^{(i)} = T_{-t,i}^{q-t}$ for every $t \in \mathbb{N}$ and $1 \leq i \leq k$.

2. DENSITY RESULTS

In this section, using Theorem 1.4, we will give, via left Følner sequences, a sufficient condition (in Theorem 2.4) in order a subset of rational numbers to contain arbitrary long arithmetic progressions, using a result of Hindman and Strauss (Theorem 2.3). Also, using Furstenberg-Katznelson's density Hales-Jewett theorem for words over a finite alphabet (Theorem 2.7), we prove in Theorem 2.8 a density result viewing the rational numbers as located words.

Firstly, we will define the left Følner sequences.

Definition 2.1. Let $(S, +)$ be a semigroup. A *left Følner sequence* in $[S]_{>0}^{\leq \omega}$ is a sequence $\{F_n\}_{n \in \mathbb{N}}$ in $[S]_{>0}^{\leq \omega}$ such that for each $s \in S$,

$$\lim_{n \rightarrow \infty} \frac{|(s + F_n) \Delta F_n|}{|F_n|} = 0,$$

where $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Remark 2.2 ([HS]). If $(S, +)$ is an infinite countable left cancellative semigroup (i.e., $a + b = a + c \Rightarrow b = c$ for every $a, b, c \in S$), then we can find a left Følner sequence in $[S]_{>0}^{\leq \omega}$.

Given a left Følner sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ in $[S]_{>0}^{\leq \omega}$, there is a natural notion of upper density associated with \mathcal{F} , namely

$$\bar{d}_{\mathcal{F}}(A) = \limsup_{n \rightarrow \infty} \frac{|A \cap F_n|}{|F_n|}.$$

In order to prove Theorem 2.4, which gives a sufficient condition in order a subset of rational numbers to contain arbitrary long arithmetic progressions, we will need some notions from the theory of ultrafilters and also Theorem 2.3, a fundamental result of Hindman and Strauss, which we mention below.

Ultrafilters. Let X be a non-empty set. An *ultrafilter* on the set X is a zero-one finite additive measure μ defined on all the subsets of X . The set of all ultrafilters on the set X is denoted by βX . So, $\mu \in \beta X$ if and only if

- (i) $\mu(A) \in \{0, 1\}$ for every $A \subseteq X$, $\mu(X) = 1$, and
- (ii) $\mu(A \cup B) = \mu(A) + \mu(B)$ for every $A, B \subseteq X$ with $A \cap B = \emptyset$.

For $\mu \in \beta X$, it is easy to see that $\mu(A \cap B) = 1$, if $\mu(A) = 1$ and $\mu(B) = 1$. For every $x \in X$ is defined the *principal ultrafilter* μ_x on X which corresponds a set $A \subseteq X$ to $\mu_x(A) = 1$ if $x \in A$ and $\mu_x(A) = 0$ if $x \notin A$. So, μ is a non-principal ultrafilter on X if and only if $\mu(A) = 0$ for every finite subset A of X .

The set βX becomes a compact Hausdorff space if it be endowed with the topology \mathfrak{T} which has basis the family $\{\bar{A} : A \subseteq X\}$, where $\bar{A} = \{\mu \in \beta X : \mu(A) = 1\}$. It is easy to see that $\overline{A \cap B} = \bar{A} \cap \bar{B}$, $\overline{A \cup B} = \bar{A} \cup \bar{B}$ and $\overline{X \setminus A} = \beta X \setminus \bar{A}$ for every $A, B \subseteq X$. We always consider the set βX endowed with the topology \mathfrak{T} . Also βX is called the *Stone-Čech compactification* of the set X .

If $(X, +)$ is a semigroup, then a binary operation $+$ is defined on βX , extending the operation $+$ on X , corresponding to every $\mu_1, \mu_2 \in \beta X$ the ultrafilter $\mu_1 + \mu_2 \in \beta X$, with

$$(\mu_1 + \mu_2)(A) = \mu_1(\{x \in X : \mu_2(\{y \in X : x + y \in A\}) = 1\}) \text{ for every } A \subseteq X.$$

With this operation $(\beta X, +, \mathfrak{T})$ becomes a right topological semigroup, that is, for every $\mu \in \beta X$ the function $f_{x_0} : \beta X \rightarrow \beta X$ with $f_{x_0}(\mu) = \mu_{x_0} + \mu$ is continuous.

Hindman and Strauss in [HS] proved the following result concerning left cancellative semigroups. For $A \subseteq S$ and $t \in S$ we set $-t + A = \{s \in S : t + s \in A\}$.

Theorem 2.3. *Let S be an infinite countable left cancellative semigroup, let $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ be a left Følner sequence in $[S]_{>0}^{\leq \omega}$, and let $A \subseteq S$. There is a countably additive measure μ on the set \mathcal{B} of Borel subsets of βS such that*

- (1) $\mu(\overline{A}) = \overline{d}_{\mathcal{F}}(A)$,
- (2) for all $B \subseteq S$, $\mu(\overline{B}) \leq \overline{d}_{\mathcal{F}}(B)$,
- (3) for all $B \in \mathcal{B}$ and all $t \in S$, $\mu(-t + B) = \mu(B) = \mu(t + B)$, and
- (4) $\mu(\beta S) = 1$.

Now we can state our theorem and give a proof in analogy to Theorem 5.6 in [HS].

Theorem 2.4. *Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ be a left Følner sequence in $[\mathbb{Q}]_{>0}^{\leq \omega}$, and let $A \subseteq \mathbb{Q}$ such that $\overline{d}_{\mathcal{F}}(A) > 0$. Then for each $k \in \mathbb{N}$ there exists $q \in \mathbb{Q}^*$ such that*

$$\overline{d}_{\mathcal{F}}(A \cap (-q + A) \cap \dots \cap (-kq + A)) > 0.$$

Proof. Let \mathcal{B} be the set of Borel subsets of $\beta\mathbb{Q}$. Pick a countably additive measure μ on \mathcal{B} which satisfies the conditions of Theorem 2.3. Let $k \in \mathbb{N}$. For $l \in \{1, \dots, k\}$ and $\nu \in \beta\mathbb{Q}$ let $T_{l,m}(\nu) = \mu_{l(-1)^{m+1}m!} + \nu$ for $m \in \mathbb{N}$ and $T_{l,m}(\nu) = \mu_{l \frac{(-1)^{-m}}{(-m+1)!}} + \nu$ for $-m \in \mathbb{N}$. Each $T_{l,n}$, $1 \leq l \leq k$, $n \in \mathbb{Z}^*$ is a continuous function from $\beta\mathbb{Q}$ to $\beta\mathbb{Q}$, since $\beta\mathbb{Q}$ is a right topological semigroup. Let $\{\mathcal{T}_l^q\}_{q \in \mathbb{Q}}$ be the rational system defined from $\{T_{l,n}\}_{n \in \mathbb{Z}^*}$ for every $1 \leq l \leq k$ respectively.

The transformations $T_{l,n}$, $1 \leq l \leq k$, $n \in \mathbb{Z}^*$ preserve μ , since μ satisfies the condition (3) of Theorem 2.3. Consequently, $\{\mathcal{T}_l^q\}_{q \in \mathbb{Q}}$, $1 \leq l \leq k$ are commuting rational systems of measure preserving transformations on the space $(\beta\mathbb{Q}, \mathcal{B}, \mu)$. According to the condition (1) of Theorem 2.3, we have $\mu(\overline{A}) = \overline{d}_{\mathcal{F}}(A) > 0$. So, using Theorem 1.4, we can find $q \in \mathbb{Q}^*$ such that

$$\mu(\overline{A} \cap (\mathcal{T}_1^q)^{-1}(\overline{A}) \cap \dots \cap (\mathcal{T}_k^q)^{-1}(\overline{A})) > 0.$$

This gives

$$\begin{aligned} \overline{d}_{\mathcal{F}}(A \cap (-q + A) \cap \dots \cap (-kq + A)) &= \overline{\mu(\overline{A} \cap (-q + A) \cap \dots \cap (-kq + A))} = \\ &= \mu(\overline{A} \cap (\mathcal{T}_1^q)^{-1}(\overline{A}) \cap \dots \cap (\mathcal{T}_k^q)^{-1}(\overline{A})) > 0, \end{aligned}$$

which finishes the proof. \square

Corollary 2.5. *Let $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ be a left Følner sequence in $[\mathbb{Q}]_{>0}^{\leq \omega}$, and let $A \subseteq \mathbb{Q}$ such that $\overline{d}_{\mathcal{F}}(A) > 0$. Then for each $k \in \mathbb{N}$ there exist $q \in \mathbb{Q}^*$ and $p \in A$ such that*

$$p + jq \in A \text{ for every } 0 \leq j \leq k.$$

Proof. Let $k \in \mathbb{N}$. According to the proof of Theorem 2.4 there exists $q \in \mathbb{Q}^*$ such that $\overline{d}_{\mathcal{F}}(A \cap (-q + A) \cap \dots \cap (-kq + A)) > 0$. Pick $p \in A \cap (-q + A) \cap \dots \cap (-kq + A)$. \square

Remarks 2.6. (1) In the statements of Theorem 2.4 and Corollary 2.5 the rational number q can be located either in \mathbb{Z}^* or in $(e^{-1} - 1, e^{-1}) \cap \mathbb{Q}^*$, using in the respective proof the results of Remark 1.5.

(2) Defining for a left Følner sequence $\mathcal{F} = \{F_n\}_{n \in \mathbb{N}}$ in $[\mathbb{Q}]_{>0}^{\leq \omega}$, and $A \subseteq \mathbb{Q}$ the density

$$d_{\mathcal{F}}^*(A) = \sup \left\{ \alpha : (\forall m \in \mathbb{N})(\exists n \geq m)(\exists q \in \mathbb{Q})(|A \cap (q + F_n)| \geq \alpha |F_n|) \right\}$$

we can replace $\bar{d}_{\mathcal{F}}$ with $d_{\mathcal{F}}^*$ in the statements of Theorem 2.4 and Corollary 2.5, using Theorem 4.6 in [HS], instead of Theorem 2.3.

Viewing the rational numbers as words and using the density Hales-Jewett theorem of Furstenberg and Katznelson ([FuKa]), we will prove in Theorem 2.8 another density result for the set of rational numbers. Let start with the necessary notation.

Let $\Sigma = \{\alpha_1, \dots, \alpha_k\}$ for $k \in \mathbb{N}$ a finite set and $v \notin \Sigma$. We denote by $W(\Sigma)$ the set of all the words $w = w_1 \dots w_n$, where $n \in \mathbb{N}$ and $w(i) \in \Sigma$ for every $1 \leq i \leq n$, and by $W(\Sigma, v)$ the set of all the (variable) words in $W(\Sigma \cup \{v\})$ with at least one occurrence of the symbol v . A *combinatorial line* in $W(\Sigma)$ is a set $\{w(\alpha) : \alpha \in \Sigma\}$ obtained by substituting the variable v of the variable word $w(v)$ by the symbols $\alpha_1, \dots, \alpha_k$. We also denote by $W_n(\Sigma)$ the subset of $W(\Sigma)$ consisting of all the words of length n .

Furstenberg and Katznelson in [FuKa2] proved the following theorem:

Theorem 2.7. *Let $\Sigma = \{\alpha_1, \dots, \alpha_k\}$, $k \in \mathbb{N}$ a finite alphabet. If $A \subseteq W(\Sigma)$ and $\limsup_n \frac{|A \cap W_n(\Sigma)|}{k^n} > 0$, then A contains a combinatorial line.*

For every $n, k \in \mathbb{N}$ and integers $t_1^n < \dots < t_n^n$ with $|t_j^n| \geq k$ for every $1 \leq j \leq n$ we define, in analogy to the representation of rational numbers, the subset $\mathbb{Q}(t_1^n, \dots, t_n^n, k) \subseteq \mathbb{Q}^*$ as

$$\begin{aligned} \mathbb{Q}(t_1^n, \dots, t_n^n, k) = \{ \sum_{j=1}^n q t_j^n c_{t_j^n} : 1 \leq q t_j^n \leq k, c_{t_j^n} = \frac{(-1)^{-t_j^n}}{(-t_j^n + 1)!} \text{ if } t_j^n < 0 \\ \text{and } c_{t_j^n} = (-1)^{t_j^n + 1} t_j^n! \text{ if } t_j^n > 0 \}. \end{aligned}$$

For $\Sigma = \{1, \dots, k\}$, $k \in \mathbb{N}$ we define

$$g : W(\Sigma) \rightarrow \bigcup_{n \in \mathbb{N}} \mathbb{Q}(t_1^n, \dots, t_n^n, k) \text{ with } g(w_1 \dots w_n) = \sum_{j=1}^n w_j c_{t_j^n}.$$

Note that $g|_{W_n(\Sigma)} : W_n(\Sigma) \rightarrow \mathbb{Q}(t_1^n, \dots, t_n^n, k)$ is 1-1 and onto for every $n \in \mathbb{N}$.

Using Theorem 2.7 we have the following density theorem:

Theorem 2.8. *Let $k \in \mathbb{N}$ and a sequence $((t_1^n, \dots, t_n^n))_{n \in \mathbb{N}}$ with $t_1^n < \dots < t_n^n$ and $|t_j^n| \geq k$ for every $1 \leq j \leq n$, $n \in \mathbb{N}$. If $A \subseteq \mathbb{Q}$ with $\limsup_n \frac{|A \cap \mathbb{Q}(t_1^n, \dots, t_n^n, k)|}{k^n} > 0$, then there exist $p \in A$ and $q \in \mathbb{Q}^*$ with $\text{dom}(p), \text{dom}(q) \subseteq \{t_1^n, \dots, t_n^n\}$ for some $n \in \mathbb{N}$, such that*

$$p + iq \in A \text{ for every } i = 0, 1, \dots, k-1.$$

Proof. Let $\Sigma = \{1, \dots, k\}$. Since $|g^{-1}(A) \cap W_n(\Sigma)| = |A \cap \mathbb{Q}(t_1^n, \dots, t_n^n, k)|$ for every $n \in \mathbb{N}$, the set $g^{-1}(A)$ contains a combinatorial line $\{w(\alpha) : \alpha \in \Sigma\}$ obtained by a variable word $w(v)$, $v \notin \Sigma$, according to Theorem 2.7. Let n be the length of $w(v)$. Then, $\{g(w(\alpha)) : \alpha \in \Sigma\} \subseteq A \cap \mathbb{Q}(t_1^n, \dots, t_n^n, k)$. So, there exist $F_1 = \{t \in \text{dom}(w) : w_t \in \Sigma\}$, $F_2 = \{t \in \text{dom}(w) : w_t = v\}$ with $F_1, F_2 \subseteq \{t_1^n, \dots, t_n^n\}$ and $F_1 \cap F_2 = \emptyset$ such that if $q = \sum_{t \in F_2} c_t$ and $p = q + \sum_{t \in F_1} w_t c_t$, where $c_t = \frac{(-1)^{-t}}{(-t+1)!}$ if $t < 0$ and $c_t = (-1)^{t+1} t!$ if $t > 0$, we have that $g(w(i)) = p + iq \in A$ for every $0 \leq i \leq k - 1$. \square

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REFERENCES

- [BIP] T. Budak, N. Işik and J. Pym, *Subsemigroups of Stone-Čech compactifications*, Math. Proc. Cambridge Phil. Soc. **116** (1994), 99–118.
- [Fu] H. Furstenberg, *Ergodic behavior of diagonal measures and a theorem of Szemerédi on arithmetic progressions*, J. d' Analyse Math. **31**, (1977), 204–256.
- [FuKa] H. Furstenberg, Y. Katznelson, *An ergodic Szemerédi theorem for IP-systems and combinatorial theory*, Journal d' Analyse Mathématique, Vol. **45** (1985) 117–168.
- [FuKa2] H. Furstenberg, Y. Katznelson, *A Density Version of the Hales-Jewett Theorem*, Journal d' Analyse Math. **57** (1991) 64–119.
- [HS] N. Hindman, D. Strauss, *Density in Arbitrary Semigroups*, Semigroup Forum Vol. **73** (2006) 273–300.
- [K] A. Koutsogiannis, *rational dynamical systems*, Topology and its Applications, **159** (2012), 1993–2003.
- [Sz] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression*, Acta Arith. **27** (1975), 199–245.
- [vdW] B. L. van der Waerden, *Beweis einer Baudetschen Vertmutung*, Nieuw Arch. Wisk. **15** (1927), 212–216.

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