CLOSEST INTEGER POLYNOMIAL MULTIPLE RECURRENCE ALONG SHIFTED PRIMES

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Abstract. Following an approach presented by N. Frantzikinakis, B. Host and B. Kra, we show that the parameters in the multidimensional Szemerédi theorem for closest integer polynomials have non-empty intersection with the set of shifted primes $\mathbb{P}-1$ (or similarly of $\mathbb{P}+1$). Using the Furstenberg Correspondence Principle, we show this result by recasting it as a polynomial multiple recurrence result in measure ergodic theory. Furthermore, we obtain integer part polynomial convergence results by the same method, which is a transference principle that enables one to deduce results for $\mathbb{Z}$-actions from results for flows. We also give some applications of our approach on Gowers uniform sets.

1. Introduction and main results

For a subset $E \subset \mathbb{Z}^\ell$ we denote its *upper Banach density* by $d^*(E)$, which is defined to be the number

$$d^*(E) = \limsup_{|I|\to\infty} \frac{|E \cap I|}{|I|},$$

where the lim sup is taken over all the parallelepipeds $I \subset \mathbb{Z}^\ell$ whose side lengths tend to infinity.

Bergelson and Leibman, in [4], proved the following polynomial multidimensional Szemerédi theorem:

Theorem. *Let $\ell, m \in \mathbb{N}$, $\vec{q}_1, \ldots, \vec{q}_m : \mathbb{Z} \to \mathbb{Z}^\ell$ be polynomials with $\vec{q}_i(0) = \vec{0}$ for $1 \leq i \leq m$ and let $E \subseteq \mathbb{Z}^\ell$ with $d^*(E) > 0$. Then there exists $n \in \mathbb{N}$ such that*

$$d^*\left(E \cap (E + \vec{q}_1(n)) \cap \ldots \cap (E + \vec{q}_m(n))\right) > 0.$$

Let $\mathbb{P}$ denote the set of prime numbers. Frantzikinakis, Host and Kra showed, in [13], that the parameters of this result can be restricted to the shifted primes $\mathbb{P}-1$ (and similarly to $\mathbb{P}+1$), generalizing results due to Sárközy ([21]), who showed that any $E-E$, with $E \subset \mathbb{Z}$, $d^*(E) > 0$, contains a shifted prime $p-1$ (and similarly a shifted prime $p+1$) for some $p \in \mathbb{P}$; due to Wooley and Ziegler ([25]) who proved the $\ell = 1$ case, and finally, of Bergelson, Leibman and Ziegler ([5]) who got the result for linear polynomials.

If $[[\cdot]]$ denotes the closest integer, i.e, $[[x]] = [x + 1/2]$, where $[\cdot]$ is the integer part function, for $\vec{q} = (q_1, \ldots, q_m)$ we set $[[\vec{q}]] = ([[q_1]], \ldots, [[q_m]])$.

Following the approach presented in [13] and methods from [19], we will prove the analogous result of Frantzikinakis, Host and Kra for the respective closest integer polynomials. Namely, we will prove the following:

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Theorem 1.1. *Let ℓ, $m \in \mathbb{N}$, $\vec{q}_1, \ldots, \vec{q}_m : \mathbb{Z} \to \mathbb{R}^\ell$ be polynomials with $\vec{q}_i(0) = \vec{0}$ for $1 \leq i \leq m$ and let $E \subseteq \mathbb{Z}^\ell$ with $d^*(E) > 0$. Then the set of integers n such that*

$$d^*\left(E \cap (E + [[\vec{q}_1(n)]]) \cap \ldots \cap (E + [[\vec{q}_m(n)]])\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

In order to obtain this result, we will make use of the Furstenberg Correspondence Principle (see below). Via this principle, we will get Theorem 1.1, by proving a multiple recurrence result in ergodic theory (see Theorem 1.2 below).

Definition. For $\ell \in \mathbb{N}$, we call the setting $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ a *system*, where $T_1, \ldots, T_\ell \colon X \to X$ are invertible commuting measure preserving transformations on the probability space $(X, \mathcal{X}, \mu)$.

Theorem (Furstenberg Correspondence Principle, [14]). *Let $\ell \in \mathbb{N}$ and $E \subseteq \mathbb{Z}^\ell$. There exist a system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and a set $A \in \mathcal{X}$ with $\mu(A) = d^*(E)$, such that*

$$d^*\left(E \cap (E + \vec{n}_1) \cap \ldots \cap (E + \vec{n}_m)\right) \geq \mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-n_{i,1}})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-n_{i,m}})A\right)$$

for all $m \in \mathbb{N}$ and $\vec{n}_j = (n_{1,j}, \ldots, n_{\ell,j}) \in \mathbb{Z}^\ell$ for $1 \leq j \leq m$.

By the Furstenberg Correspondence Principle, in order to prove Theorem 1.1, it suffices to prove the following ergodic reformulation of it:

Theorem 1.2. *Let $\ell, m \in \mathbb{N}$ and $q_{i,j} \in \mathbb{R}[t]$ be polynomials with $q_{i,j}(0) = 0$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$. Then, for every system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and any $A \in \mathcal{X}$ with $\mu(A) > 0$, the set of integers n such that*

$$\mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]})A\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

Remark. As in [13], the arguments will show that the aforementioned intersection has positive measure for a set of positive relative density in the shifted primes (and so, the same holds for the conclusion of Theorem 1.1 as well).

Our method will show though that we can have integer part polynomial multiple convergence along shifted primes. Namely, we prove:

Theorem 1.3. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $q_{i,j} \in \mathbb{R}[t]$ polynomials, $1 \leq i \leq \ell$, $1 \leq j \leq m$ and $f_1, \ldots, f_m \in L^\infty(\mu)$ be functions with $\|f_i\|_\infty \leq 1$. Then the averages*

$$\frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} (\prod_{i=1}^{\ell} T_i^{[q_{i,1}(p)]})f_1 \cdot \ldots \cdot (\prod_{i=1}^{\ell} T_i^{[q_{i,m}(p)]})f_m$$

converge in $L^2(\mu)$ as $N \to \infty$, where $\pi(N) = |\mathbb{P} \cap [1,N]|$ denotes the number of primes up to N and $[1,N]$ denotes the set $\{1, \ldots, N\}$.

So, we generalize Theorem 1.3 from [13] for integer part polynomial iterates and Theorem 1.1 from [22] for general polynomials and commuting transformations. Also, since $[[x]] = [x + 1/2]$, Theorem 1.3, holds for closest integer polynomial iterates as well.

Remark. We cannot in general obtain polynomial multiple recurrence results with iterates given by integer part polynomials (i.e, the conclusion of Theorem 1.2, and consequently the conclusion of Theorem 1.1). Indeed, since $[-x] = -[x] - 1$ when $x \notin \mathbb{Z}$, $x < 0$, for $a \notin \mathbb{Q}$, we can find examples with $\mu(T^{[an]}A \cap T^{[-an]}A) = 0$ for every $n \in \mathbb{N}$.

Frantzikinakis, Host and Kra, in [12], first showed, using Theorem 2.2 for $d = 3$ (see below-this result was a conjecture, the Inverse Conjecture for Gowers Norms, at that time for $d \geq 4$), the respective results of Theorem 1.3 and Theorem 1.2 (for the second one see the proof of Theorem 1.2 and combine it with Lemma 2.1 below) in Theorems 3 and 4 respectively, for $\ell = 2$ commuting transformations (see Section 7 of [12]) and $m = 2$ linear iterates of the form n and $2n$.

Wooley and Ziegler ([25]), showed (in [25, Theorem 1.1]) Theorem 1.1 for the special case of $\ell = 1$ and integer valued polynomials with no constant terms. Also, in [25, Theorem 1.2] they obtained, for a single transformation and integer valued polynomial iterates with no constant terms, exploiting the PET induction scheme from [1], the respective result of Theorem 1.2.

The same authors of [12], in [13], with similar arguments from [12], using the full strength of Theorem 2.2 and the PET induction scheme from [1], showed, in [13, Theorem 1.3] and [13, Theorem 1.2], the respective results of Theorem 1.3 and Theorem 1.2 for any integer valued polynomial (with no constant terms on the polynomial iterates in the second case). [13, Theorem 1.2] via Furstenberg Correspondence Principle gives [13, Theorem 1.1] which is the respective result of Theorem 1.1 for vectors of integer valued polynomials with no constant terms.

W. Sun, in [22], first obtained results for integer part linear iterates. More specifically, [22] treated Theorems 1.1, 1.2 and 1.3, for $\ell = 1$ and iterates of the form $[j\alpha n]$ for $1 \leq j \leq m$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$, or of the form $j[\alpha n]$ for $1 \leq j \leq m$ and $\alpha \in \mathbb{R} \setminus \mathbb{Q}$. The proofs of these results will be simplified and extended by our method in the more general setting of commuting transformations. More specifically, the respective result of Theorem 1.3 ([22, Theorem 1.1]), follows immediately from the linear case of Theorem 1.3. The respective result of Theorem 1.2, and so, of Theorem 1.1 ([22, Theorem 1.2] and [22, Corollary 1.3] respectively), follows from the Theorem 1.4, which is a multiple recurrence result on integer part monomials with specific coefficients:

Theorem 1.4. *Let $\ell, m \in \mathbb{N}$, $a \in \mathbb{R}$, $d_{i,j} \in \mathbb{N}$ positive integers and $k_{i,j} \in \mathbb{N} \cup \{0\}$ be non-negative integers for $1 \leq i \leq \ell$, $1 \leq j \leq m$. Then, for every system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ and any $A \in \mathcal{X}$ with $\mu(A) > 0$, the set of integers n such that*

$$\mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[ak_{i,1}n^{d_{i,1}}]})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[ak_{i,m}n^{d_{i,m}}]})A\right) > 0$$

has non-empty intersection with $\mathbb{P} - 1$ and $\mathbb{P} + 1$.

As we have already mentioned, in general, our methods follow the approach of Frantzikinakis, Host and Kra ([12] and [13]) using a transference principle (used in [19] as well) that allows one to deduce results for integer part (and hence for closest integer) polynomial iterates, with real coefficients, from respective results for integer valued polynomial iterates. Also, in order to obtain the respective uniform multiple recurrence result for closest integer polynomial iterates, in Proposition 3.8, which is a crucial intermediate step for proving Theorem 1.2, we use a method presented in [2] and used in [22] as well.

Finally, we give an application of our approach on some recent work of Franzikinakis and Host ([11]) on Gowers uniform sets. More specifically, we prove (see Section 5 for definitions and details):

Theorem 1.5. *Any shift of a Gowers uniform set $S \subseteq \mathbb{N}$ is a set of closest integer polynomial multiple recurrence and integer part polynomial multiple mean convergence.*

At this point we will borrow some examples from [11] of Gowers uniform sets, and so, sets for which we have the conclusion of Theorem 1.5.

Remarks ([11]). If $\omega(n)$ is the number of distinct prime factors of an integer n and $\Omega(n)$ the number of prime factors of n counted with multiplicity, then:

i) Any shift of the sets

$$S_{\omega,A,b} := \{n \in \mathbb{N} :\ \omega(n) \equiv a(\mathrm{mod} b) \textit{ for some } a \in A\},$$

and similarly $S_{\Omega,A,b}$, for every $b \in \mathbb{N}$ and $A \subseteq \{0, \dots, b-1\}$ non-empty, is Gowers uniform.

ii) Any shift of the sets

$$S_{\omega,A,\alpha} := \{n \in \mathbb{N} :\ \|\omega(n)\alpha\| \in A\},$$

and similarly $S_{\Omega,A,\alpha}$, for every irrational α and Riemann-measurable set $A \subseteq [0, 1/2]$ of positive measure, where $\|x\| := d(x, \mathbb{Z})$ denotes the distance to the closest integer, is Gowers uniform.

iii) The same results to ii) hold if $S_{\omega,A,\alpha}$ and $S_{\Omega,A,\alpha}$ are defined using fractional parts.

Even if it's not stated, without loss of generality, for the bounded functions f_i, which appear in our expressions, we will always assume that $\|f_i\|_\infty \leq 1$ for all i.

Notation. We always denote by $[\cdot]$, $\{\cdot\}$, $[[\cdot]]$ and $\|\cdot\|$ the integer part, the fractional part, the closest integer and the distance to the closest integer function respectively. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{Z}_N = \mathbb{Z}/N\mathbb{Z}$ the integers modulo N. When we need, we identify the set $\mathbb{Z}_N$ with the set $[1, N]$. For a finite set F and $a : F \to \mathbb{C}$, we write $\mathbb{E}_{n\in F} a(n) = \dfrac{1}{|F|} \sum_{n\in F} a(n)$. For a measurable function f on a measure space X with a transformation $T : X \to X$, we denote with Tf the composition $f \circ T$. Given transformations $T_i : X \to X$, $1 \leq i \leq \ell$, with $\prod_{i=1}^{\ell} T_i$ we denote the composition $T_1 \circ \cdots \circ T_\ell$. With (a, b) we denote the greatest common divisor of a, b. A quantity that goes to 0 as $N \to \infty$ is denoted as $o_N(1)$, while a quantity that goes to 0 as $N \to \infty$ and then $w \to \infty$ as $o_{N\to\infty;w}(1)$.

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2. Definitions and tools

In this section we give the definitions and the main ideas in order to prove Theorem 1.2. For reader's convenience, we repeat most of the part of Section 2 from [13].

We start by recalling the definition of the *von Mangoldt function*, $\Lambda : \mathbb{N} \to \mathbb{R}$, where $\Lambda(n) = \begin{cases} \log(p) & , \text{ if } n = p^k \text{ for some } p \in \mathbb{P} \text{ and some } k \in \mathbb{N} \\ 0 & , \text{ elsewhere} \end{cases}.$

As in [13], it is more natural for us to deal, in stead of Λ, with the function $\Lambda' : \mathbb{N} \to \mathbb{R}$, where $\Lambda'(n) = \mathbf{1}_{\mathbb{P}}(n) \cdot \Lambda(n) = \mathbf{1}_{\mathbb{P}}(n) \cdot \log(n)$.

The function Λ', according to the following lemma, will allow us to relate averages along primes with weighted averages over the integers.

Lemma 2.1 ([12])**.** *If $a : \mathbb{N} \to \mathbb{C}$ is bounded, then*

$$\Big| \frac{1}{\pi(N)} \sum_{p \in \mathbb{P} \cap [1,N]} a(p) - \frac{1}{N} \sum_{n=1}^{N} \Lambda'(n) \cdot a(n) \Big| = o_N(1).$$

Next, we recall the definition of Gowers norms.

Definition. If $a : \mathbb{Z}_N \to \mathbb{C}$, we inductively define:

$$\|a\|_{U_1(\mathbb{Z}_N)} = \Big| \mathbb{E}_{n \in \mathbb{Z}_N} a(n) \Big|;$$

and

$$\|a\|_{U_{d+1}(\mathbb{Z}_N)} = \Big(\mathbb{E}_{h \in \mathbb{Z}_N} \|a_h \cdot \overline{a}\|^{2^d}_{U_d(\mathbb{Z}_N)} \Big)^{1/2^{d+1}},$$

where $a_h(n) = a(n+h)$. As Gowers showed in [15], $\|\cdot\|_{U_d(\mathbb{Z}_N)}$ defines a norm on $\mathbb{Z}_N$ for $d \geq 2$.

For $w > 2$, let

$$W = \prod_{p \in \mathbb{P} \cap [1, w-1]} p$$

be the product of primes bounded above by w. For $r \in \mathbb{N}$, let

$$\Lambda'_{w,r}(n) = \frac{\phi(W)}{W} \cdot \Lambda'(Wn + r),$$

where ϕ is the Euler function, be the *modified von Mangoldt function.*

The next result, which will play an important role in our proof, shows the Gowers uniformity of the modified von Mangoldt function and can be derived by [16], [17] and [18]. We write, and we will later use, the formulation that can be found in [13].

Theorem 2.2 ([16, Theorem 7.2]). *For every $d \in \mathbb{N}$ we have that*

$$\lim_{w\to\infty}\left(\lim_{N\to\infty}\left(\max_{1\leq r\leq W,\,(r,W)=1}\left\|(\Lambda'_{w,r}-1)\cdot\mathbf{1}_{[1,N]}\right\|_{U_d(\mathbb{Z}_{dN})}\right)\right)=0.$$

The following uniformity estimate, is an important step towards our result. It will allow us to obtain a similar estimate in our setting, in Theorem 3.1, in order to make use of the conclusion of Theorem 2.2.

Lemma 2.3 ([13, Lemma 3.5]). *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ be a system, $q_{i,j} \in \mathbb{Z}[t]$ polynomials, $1 \leq i \leq \ell$, $1 \leq j \leq m$, $f_1, \dots, f_m \in L^\infty(\mu)$ and $a : \mathbb{N} \to \mathbb{C}$ be a sequence satisfying $a(n)/n^c \to 0$ for every $c > 0$. Then there exists $d \in \mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ, m and a constant C_d depending on d, such that*

$$\left\|\frac{1}{N}\sum_{n=1}^{N}a(n)\cdot(\prod_{i=1}^{\ell}T_i^{q_{i,1}(n)})f_1\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{q_{i,m}(n)})f_m\right\|_{L^2(\mu)}\leq C_d\left(\left\|a\cdot\mathbf{1}_{[1,N]}\right\|_{U_d(\mathbb{Z}_{dN})}+o_N(1)\right).$$

Furthermore, the constant C_d is independent of the sequence $(a(n))$ and the $o_N(1)$ term depends only on the integer d and on the sequence $(a(n))$.

In order to prove this result, the authors in [13], successively made use of the following variant of van der Corput's estimate:

Lemma 2.4. *Let $N \in \mathbb{N}$ and $v(1), \dots, v(N)$ be elements in an inner product space. Then there exists a constant C such that*

$$\left\|\frac{1}{N}\sum_{n=1}^{N}v(n)\right\|^2\leq C\left(\frac{1}{N^2}\sum_{n=1}^{N}\|v(n)\|^2+\frac{1}{N}\sum_{h=1}^{N}\left|\frac{1}{N}\sum_{n=1}^{N-h}\langle v(n+h),v(n)\rangle\right|\right).$$

Remark. In order to prove Lemma 2.3, we successively use Lemma 2.4, using the van der Corput operation, choosing every time appropriate polynomials in order to have reduction in the complexity. Actually, the number d that Lemma 2.3 provides, corresponds to the $d-1$ steps we need to do in order the polynomials to be reduced into constant ones, by using the Polynomial Exhaustion Technique (PET) induction, introduced in [1].

For more information and details on the van der Corput operation and the scheme of the PET induction we are using here, we refer the reader to [13].

3. Main estimates

In this section we prove some key estimates needed in the proof of Theorems 1.2, 1.3, 1.4 and 1.5. The main idea is to use a transference principle that allows to deduce results for integer part (and hence for closest integer) polynomial iterates with real coefficients from respective results for polynomial iterates with integer coefficients.

We first recall the definition of a measure preserving flow.

Definition. Let $r \in \mathbb{N}$ and $(X, \mathcal{X}, \mu)$ be a probability space. We call a jointly measurable family $(T_t)_{t\in\mathbb{R}^r}$ of measure preserving transformations $T_t \colon X \to X$, a *measure preserving flow*, if it satisfies

$$T_{s+t} = T_s \circ T_t$$

for all $s, t \in \mathbb{R}^r$.

Also, we recall the notion of the *lower* and *upper density* for a subset $S \subseteq \mathbb{N}$ to be the number $\underline{d}(S)$ and $\overline{d}(S)$ respectively, where

$$\underline{d}(S) = \liminf_{N\to\infty} \frac{|S \cap [1, N]|}{N} \quad \text{and} \quad \overline{d}(S) = \limsup_{N\to\infty} \frac{|S \cap [1, N]|}{N}.$$

In case $\overline{d}(S) = \underline{d}(S)$, we call the common value *density* of S.

The following important remarks contain essential information that we will use in various proofs along this article.

Remarks. i) Let $\ell, m \in \mathbb{N}$ and $q_{i,j} \in \mathbb{R}[t]$ be real valued polynomials for $1 \leq i \leq \ell$, $1 \leq j \leq m$. Then, for any $\mathbb{R}^{\ell m}$ measure preserving flow $\prod_{i=1}^{\ell} T_{i,s_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,s_{i,m}}$, where the transformations $T_{i,s_{i,j}}$ are defined in the probability space $(X, \mathcal{X}, \mu)$, $s_{i,j} \in \mathbb{R}$ and functions $f_1, \ldots, f_m \in L^\infty(\mu)$, the sequence of functions

$$\tilde{b}(n) = (\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot q_{i,1}(n)})f_1 \cdot \ldots \cdot (\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot q_{i,m}(n)})f_m,$$

where $\delta_{ij} = \begin{cases} 0 & , i \neq j \\ 1 & , i = j \end{cases}$ is the Kronecker delta function (note that $T_{i,0}$ is the identity for all i) satisfies the conclusion of Lemma 2.3, with $\tilde{b}(n)$ in place of the sequence

$$(\prod_{i=1}^{\ell} T_i^{q_{i,1}(n)})f_1 \cdot \ldots \cdot (\prod_{i=1}^{\ell} T_i^{q_{i,m}(n)})f_m.$$

Indeed, if $q(t) = a_r t^r + \ldots + a_1 t + a_0 \in \mathbb{R}[t]$, we write $T_{q(t)} = (T_{a_r})^{n^r} \cdot \ldots \cdot (T_{a_1})^n \cdot T_{a_0}$ and we use Lemma 2.3 for the invertible commuting measure preserving transformations $S_1 = T_{a_1}, \ldots, S_r = T_{a_r}$.

ii) Real valued polynomials, $q \in \mathbb{R}[t]$, satisfy the condition:

$$\lim_{\delta\to 0^+} \overline{d}\left(\left\{n \in \mathbb{N} \colon \{q(n)\} \in [1-\delta, 1)\right\}\right) = 0,$$

where $\{\cdot\}$ denotes the fractional part.

Indeed, let $q(t) = a_r t^r + \ldots + a_1 t + a_0 \in \mathbb{R}[t]$.
If $a_i \notin \mathbb{Q}$ for some $1 \leq i \leq r$, then we have the condition from Weyl's result, since $(q(n))$ is uniformly distributed (mod1).
If $a_i \in \mathbb{Q}$ for all $1 \leq i \leq r$, then the sequence $(q(n))$ is periodic (mod1) and the conclusion is obvious.

iii) If f is Riemann-integrable on $[0, 1)$ with $\int_{[0,1)} f(x)\, dx = c$, then for every $\varepsilon > 0$ we can find trigonometric polynomials $q_1,\ q_2$, with no constant terms, in order to have the relation

$$q_1(t) + c - \varepsilon \leq f(t) \leq q_2(t) + c + \varepsilon.$$

The following result is proved via a transference principle that enables one to deduce results for $\mathbb{Z}$-actions from results for flows (see also [19] in comparison with [9]). This technique was first used in [20] by E. Lesigne, in order to prove that when a sequence of real positive numbers is good for the single term pointwise ergodic theorem, then the respective sequence of its integer parts is also good (see [7] as well). This method was later used by M. Wierdl (in [24]) to deal with multiple term averages (see Theorem 3.2 in [19]).

Theorem 3.1. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system, $q_{i,j} \in \mathbb{R}[t]$ polynomials, $1 \leq i \leq \ell$, $1 \leq j \leq m$ and $f_1, \ldots, f_\ell \in L^\infty(\mu)$.*

For the sequence of functions

$$b(n) = (\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]})f_1 \cdot \ldots \cdot (\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]})f_m$$

there exists $d \in \mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, such that for every $0 < \delta < 1$ there exists a constant $C_{d,\delta}$ depending on d and δ, such that

$$\begin{aligned} \left\| \frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{w,r}(n) - 1)b(n) \right\|_{L^2(\mu)} &\leq C_{d,\delta}\left(\left\|(\Lambda'_{w,r} - 1)\cdot \mathbf{1}_{[1,N]}\right\|_{U_d(\mathbb{Z}_{dN})} + o_N(1) \right) \\ &+ c_\delta\left(1 + o_{N\to\infty;w}(1)\right), \end{aligned}$$

for all $r \in \mathbb{N}$, where $c_\delta \to 0$ as $\delta \to 0^+$ and the term $o_N(1)$ depends on the integer d.

Remark. Due to unpleasant error terms, it seems difficult to adapt the PET induction used in the proof of Lemma 2.3 in order to prove the asserted estimate.

Proof of Theorem 3.1. Let $0 < \delta < 1$ and $w, r \in \mathbb{N}$. For the given transformations on X, we define the $\mathbb{R}^{\ell m}$ action $\prod_{i=1}^{\ell} T_{i,s_{i,1}} \cdot \ldots \cdot \prod_{i=1}^{\ell} T_{i,s_{i,m}}$ on the probability space $Y = X \times [0,1)^{\ell m}$, endowed with the measure $\nu = \mu \times \lambda^{\ell m}$ (λ is the Lebesgue measure on $[0,1)$), by

$$\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,s_{i,j}}(x, a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,2}, \ldots, a_{1,m}, \ldots, a_{\ell,m}) =$$

$$\left(\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_i^{[s_{i,j}+a_{i,j}]}x, \{s_{1,1}+a_{1,1}\}, \ldots, \{s_{\ell,1}+a_{\ell,1}\}, \ldots, \{s_{1,m}+a_{1,m}\}, \ldots, \{s_{\ell,m}+a_{\ell,m}\} \right).$$

Since the transformations $T_1, \ldots, T_\ell$ are measure preserving and commute, and also since $[x+\{y\}]+[y] = [x+y]$, it is easy to check that the above action defines a measure preserving flow on the product probability space Y.

If $f_1, \ldots, f_m$ are bounded functions on X, we define the Y-extensions of f_j, setting for every element $(a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,2}, \ldots, a_{1,m}, \ldots, a_{\ell,m}) \in [0,1)^{\ell m}$:

$$\hat{f}_j(x, a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,2}, \ldots, a_{1,m}, \ldots, a_{\ell,m}) = f_j(x), \quad 1 \leq j \leq m;$$

and

$$\hat{f}_0(x, a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,m}) = 1_{[0,\delta]^{\ell m}}(a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,m}).$$

If for every $n \in \mathbb{N}$ (see Remark i) before Theorem 3.1)

$$\tilde{b}(n) = \hat{f}_0 \cdot (\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{j1}\cdot q_{i,1}(n)})\hat{f}_1 \cdot \ldots \cdot (\prod_{j=1}^{m}\prod_{i=1}^{\ell} T_{i,\delta_{jm}\cdot q_{i,m}(n)})\hat{f}_m,$$

for every $x \in X$ we define

$$b'(n)(x) = \int_{[0,1)^{\ell m}} \tilde{b}(n)(x, a_{1,1}, \ldots, a_{\ell,1}, a_{1,2}, \ldots, a_{\ell,2}, \ldots, a_{1,m}, \ldots, a_{\ell,m})\, d\lambda^{\ell m},$$

where the integration is with respect to the variables $a_{i,j}$.

Then, by using the triangle and the Cauchy-Schwarz inequality, if $a(n) = \Lambda'_{w,r}(n) - 1$, we have that

$$\delta^{\ell m} \left\| \frac{1}{N}\sum_{n=1}^{N} a(n)b(n) \right\|_{L^2(\mu)} \leq \left\| \frac{1}{N}\sum_{n=1}^{N} a(n)\cdot(\delta^{\ell m}b(n) - b'(n)) \right\|_{L^2(\mu)} + \left\| \frac{1}{N}\sum_{n=1}^{N} a(n)\tilde{b}(n) \right\|_{L^2(\nu)}.$$

From Part i) of the previous remark, we can use Lemma 2.3 to find an integer $d \in \mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ, m and a constant C_d depending on d, such that

$$\left\| \frac{1}{N}\sum_{n=1}^{N} a(n)\tilde{b}(n) \right\|_{L^2(\nu)} \leq C_d \left(\|a \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1) \right),$$

where the $o_N(1)$ term depends only on the integer d and the sequence $(a(n))$.

Next we will study the term $\left\| \frac{1}{N}\sum_{n=1}^{N} a(n)\cdot(\delta^{\ell m}b(n) - b'(n)) \right\|_{L^2(\mu)}$.

For every $x \in X$ and $n \in \mathbb{N}$, we have

$$\left|\delta^{\ell m}b(n)(x) - b'(n)(x)\right| = \Big| \int_{[0,\delta]^{\ell m}} \left(\prod_{j=1}^{m} f_j(\prod_{i=1}^{\ell} T_i^{[q_{i,j}(n)]}x) - \prod_{j=1}^{m} f_j(\prod_{i=1}^{\ell} T_i^{[q_{i,j}(n)+a_{i,j}]}x) \right) d\lambda^{\ell m} \Big|.$$

Since all the relevant $a_{i,j}$ in the integrand are less or equal than δ, if the fractional part of all $q_{i,j}(n)$ is less than $1-\delta$, we have $T_i^{[q_{i,j}(n)+a_{i,j}]} = T_i^{[q_{i,j}(n)]}$ for all $1 \leq i \leq \ell$, $1 \leq j \leq m$. We deal with the case where the fractional part of some $q_{i,j}(n)$ is greater or equal to $1-\delta$.

For every $1 \leq i \leq \ell$, $1 \leq j \leq m$, let $E_\delta^{i,j} := \{n \in \mathbb{N} \colon \{q_{i,j}(n)\} \in [1-\delta, 1)\}$.

Then, by using the fact that $\mathbf{1}_{E_\delta^{1,1}\cup\ldots\cup E_\delta^{1,m}\cup E_\delta^{2,1}\cup\ldots\cup E_\delta^{\ell,m}} \leq \sum_{(i,j)\in[1,\ell]\times[1,m]} \mathbf{1}_{E_\delta^{i,j}}$ and that $\mathbf{1}_{E_\delta^{i,j}}(n) = \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\})$, for $1 \leq i \leq \ell$, $1 \leq j \leq m$, $n \in \mathbb{N}$, for every $x \in X$ we have

$$\left|\delta^{\ell m}b(n)(x) - b'(n)(x)\right| \leq 2\delta^{\ell m} \cdot \sum_{(i,j)\in[1,\ell]\times[1,m]} \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\}),$$

so, recalling that $a(n) = \Lambda'_{w,r}(n) - 1$,

$$\frac{1}{N}\sum_{n=1}^{N} |a(n)| \cdot \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\}) \leq \frac{1}{N}\sum_{n=1}^{N} \Lambda'_{w,r}(n)\cdot \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\}) + \frac{|E_\delta^{i,j} \cap [1,N]|}{N}.$$

From Part ii) of the previous remark, we have that for small enough δ, the term (and the sum of finitely many terms of this form) $\frac{|E_\delta^{i,j}\cap[1,N]|}{N}$ is as small as we want.

As for the term $\frac{1}{N}\sum_{n=1}^N \Lambda'_{w,r}(n)\cdot \mathbf{1}_{[1-\delta,1)}(\{q_{i,j}(n)\})$, we will show that it goes to zero when $N\to\infty$, then $w\to\infty$ and finally $\delta\to 0^+$.

If the polynomial $q_{i,j}(n)$ has only (except maybe the constant term) rational coefficients, for small δ, the sum will go to zero from periodicity, and so, we can assume that the polynomial has at least one irrational coefficient (except the constant term). From Part iii) of the previous remark (for the function $f=\mathbf{1}_{[1-\delta,1)}$), it suffices to show that $\frac{1}{N}\sum_{n=1}^N \Lambda'_{w,r}(n)e^{2\pi i k q_{i,j}(n)}\to 0$ as $N\to\infty$ and then $w\to\infty$ for all $k\in\mathbb{Z}\setminus\{0\}$.

We write

$$\frac{1}{N}\sum_{n=1}^N \Lambda'_{w,r}(n)e^{2\pi i k q_{i,j}(n)}=\frac{1}{N}\sum_{n=1}^N (\Lambda'_{w,r}(n)-1)e^{2\pi i k q_{i,j}(n)}+\frac{1}{N}\sum_{n=1}^N e^{2\pi i k q_{i,j}(n)}.$$

The first term goes to zero as $N\to\infty$ and then $w\to\infty$ from [16], while the second term goes to zero as $N\to\infty$ from Weyl's equidistribution theorem. The result now follows. $\square$

Remark. The crucial fact for the proof of Theorem 3.1, is that the integer part multicorrelation sequence $(b(n))$ can be written as

$$b(n)=\frac{1}{\delta^{\ell m}}b'(n)+\frac{1}{\delta^{\ell m}}(\delta^{\ell m}b(n)-b'(n)),\ \text{ for any } \delta>0,$$

i.e., as a sum of a conditional expectation of a usual multicorrelation sequence and an error function, controlled by terms of the form $\mathbf{1}_{\{n\in\mathbb{N}:\, q(n)\in[1-\delta,1)\}}$.

Using first Theorem 3.1 and then Theorem 2.2, we will now prove a result, in Proposition 3.2 below, which will give us a comparison between averages over primes (via the modified von Mangoldt function) and averages over integers. This result, together with a uniform multiple recurrence result, that we prove in Corollary 3.9 below, reflect the main arguments for proving Theorem 1.2. The proof of Theorem 1.2 will actually make use of the Proposition 3.2 for the respective closest integer polynomial iterates. Corollary 3.9 will mainly follow by a uniform multiple recurrence result for polynomial iterates, which follows from the polynomial Szemerédi theorem (from [4]) in the same way as Theorem 2.1 (ii) and (iii) is proved in [3], in order to obtain the respective uniform multiple recurrence result for closest integer polynomial iterates, in Proposition 3.8, the proof of which is using a method presented in [2] and used in [22] as well.

Proposition 3.2. *Let $\ell,m\in\mathbb{N}$, $(X,\mathcal{X},\mu,T_1,\dots,T_\ell)$ be a system, $q_{i,j}\in\mathbb{R}[t]$ polynomials, $1\le i\le\ell$, $1\le j\le m$ and $f_1,\dots,f_\ell\in L^\infty(\mu)$.*

Then,

$$\max_{1\le r\le W,\,(r,W)=1}\left\|\frac{1}{N}\sum_{n=1}^N(\Lambda'_{w,r}(n)-1)\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(Wn+r)]})f_1\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(Wn+r)]})f_m\right\|_{L^2(\mu)}$$

converges to 0 as $N\to\infty$ and then $w\to\infty$.

Proof. Using Theorem 3.1, for the polynomials $q_{i,j}(Wn+r)$, we get that for every $0<\delta<1$, there exists $d\in\mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, and a constant $C_{d,\delta}$ depending on d and δ, such that

$$\max_{1\leq r\leq W,\ (r,W)=1}\left\|\frac{1}{N}\sum_{n=1}^{N}(\Lambda'_{w,r}(n)-1)\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(Wn+r)]})f_1\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(Wn+r)]})f_m\right\|_{L^2(\mu)}$$
$$\leq C_{d,\delta}\left(\max_{1\leq r\leq W,\ (r,W)=1}\left\|(\Lambda'_{w,r}-1)\cdot\mathbf{1}_{[1,N]}\right\|_{U_d(\mathbb{Z}_{dN})}+o_N(1)\right)+c_\delta\left(1+o_{N\to\infty;w}(1)\right),$$

where $c_\delta\to 0$ as $\delta\to 0^+$. Taking first $N\to\infty$ and then $w\to\infty$ in this expression, by Theorem 2.2, we have that the required limit is bounded above by c_δ. Taking $\delta\to 0^+$, we get the result. □

We will make use of the following uniform multiple recurrence result; it follows from the polynomial Szemerédi ([4]) in the same way as Theorem 2.1 (ii) is proved in [3]:

Theorem 3.3 ([3])**.** *Let $\ell,m\in\mathbb{N}$ and $(X,\mathcal{X},\mu,(T_{i,j})_{1\leq i\leq\ell,1\leq j\leq m})$ be a system. Then for any $A\in\mathcal{X}$ with $\mu(A)>0$, there exists a positive constant $c\equiv c_{\ell,m,\mu(A)}>0$ such that*

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\mu\left(A\cap(\prod_{i=1}^{\ell}T_{i,1}^{-n^i})A\cap\ldots\cap(\prod_{i=1}^{\ell}T_{i,m}^{-n^i})A\right)\geq c. \tag{1}$$

Remark. In fact, it is known that the limit in (1) exists from [23].

Lemma 3.4. *Let $\ell,m\in\mathbb{N}$. For every $\delta>0$ there exists a constant $c\equiv c_{\ell,m,\delta}>0$ such that:*

$$\underline{d}\left(\left\{n\in\mathbb{N}:\ \left\|a_{i,j}n^i\right\|<\delta\ \ \forall\,1\leq i\leq\ell,\ 1\leq j\leq m\right\}\right)\geq c,$$

for all $a_{i,j}\in\mathbb{R}$, $1\leq i\leq\ell$, $1\leq j\leq m$.

Proof. Let $0<\delta\leq 1/2$. For the system $(\mathbb{T},\mathcal{B}(\mathbb{T}),m,(T_{i,j})_{1\leq i\leq\ell,1\leq j\leq m})$, where

$$T_{i,j}x=x+a_{i,j}(\mathrm{mod}1),\ \ x\in\mathbb{T},\ \ 1\leq i\leq\ell,\ 1\leq j\leq m$$

and the set $A=[0,\delta)$, use Theorem 3.3 to find a constant $c\equiv c_{\ell,m,\delta}>0$ so that

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}m\left(A\cap(\prod_{i=1}^{\ell}T_{i,1}^{-n^i})A\cap\ldots\cap(\prod_{i=1}^{\ell}T_{i,m}^{-n^i})A\right)\geq c.$$

To obtain the result, note that if $x\in A\cap(\prod_{i=1}^{\ell}T_{i,1}^{-n^i})A\cap\ldots\cap(\prod_{i=1}^{\ell}T_{i,m}^{-n^i})A$, then

$$x+a_{i,j}n^i(\mathrm{mod}1)\in[0,\delta),\ \ \forall\,1\leq i\leq\ell,\ 1\leq j\leq m, \tag{2}$$

and since $x\in[0,\delta)$ itself, (2) gives us that

$$\{a_{i,j}n^i\}\in[0,\delta-x)\cup[1-x,1),\ \ \forall\,1\leq i\leq\ell,\ 1\leq j\leq m,$$

from which we have that

$$\left\|a_{i,j}n^i\right\|<\delta\ \ \forall\,1\leq i\leq\ell,\ 1\leq j\leq m,$$

hence the result. □

Remark. The proof of Lemma 3.4 can also follow by a single recurrence argument as well.

We will also need the following lemma.

Lemma 3.5. *Let $q(n) = a_1n + \ldots + a_kn^k \in \mathbb{R}[n]$, be a real valued polynomial with no constant term and for any $r \in \mathbb{N}$ let*

$$S_r := \Big\{m \in \mathbb{N} : \ \|a_im^i\| < \frac{1}{2kr^k},\ 1 \leq i \leq k\Big\}.$$

Then, for any $m \in S_r$ and $1 \leq n \leq r$ we have

$$[[q(mn)]] = [[a_1m]]n + \ldots + [[a_km^k]]n^k.$$

Proof. If $\|x_1\|+\ldots+\|x_k\| < 1/2$, we can easily check that $[[x_1+\ldots+x_k]] = [[x_1]]+\ldots+[[x_k]]$, obtaining the conclusion of the lemma. □

We will also use the following uniform multiple recurrence result which follows from the polynomial Szemerédi ([4]) in the same way as Theorem 2.1 (iii) is proved in [3]:

Theorem 3.6 ([3]). *Let $\ell, m \in \mathbb{N}$ and $(X, \mathcal{X}, \mu, (T_{i,j})_{1\leq i\leq \ell, 1\leq j\leq m})$ be a system. Then for any $A \in \mathcal{X}$ with $\mu(A) > 0$, there exist a constant $c \equiv c_{\ell,m,\mu(A)} > 0$ and an integer $N_0 \equiv N_0(\ell, m, \mu(A)) \in \mathbb{N}$ such that for some $1 \leq n \leq N_0$ we have*

$$\mu\left(A \cap (\prod_{i=1}^{\ell} T_{i,1}^{-n^i})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_{i,m}^{-n^i})A\right) \geq c. \tag{3}$$

Using Theorem 3.6, we get the following:

Proposition 3.7. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ be a system and $q_{i,j} \in \mathbb{Z}[t]$ polynomials with maximum degree d and $q_{i,j}(0) = 0$ for any $1 \leq i \leq \ell, 1 \leq j \leq m$. Then, for any $A \in \mathcal{X}$ with $\mu(A) > 0$ there exist a constant $c \equiv c_{d,m,\mu(A)} > 0$ and an integer $N_0 \equiv N_0(\ell, m, \mu(A)) \in \mathbb{N}$ such that for some $1 \leq n \leq N_0$ we have*

$$\mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-q_{i,1}(n)})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-q_{i,m}(n)})A\right) \geq c. \tag{4}$$

Proof. If $q_{i,j}(n) = a_{1,i,j}n + \ldots + a_{d,i,j}n^d$ for any $1 \leq i \leq \ell, 1 \leq j \leq m$ (we put some zero terms if needed), set

$$S_{k,j} = \prod_{i=1}^{\ell} T_i^{a_{k,i,j}},\ \ 1 \leq k \leq d,\ 1 \leq j \leq m,$$

and use Theorem 3.6 for the system $(X, \mathcal{X}, \mu, (S_{k,j})_{1\leq k\leq d, 1\leq j\leq m})$. □

Proposition 3.8. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$ a system and $q_{i,j} \in \mathbb{R}[t]$ with maximum degree d and $q_{i,j}(0) = 0$ for any $1 \leq i \leq \ell, 1 \leq j \leq m$. Then, for any $A \in \mathcal{X}$ with $\mu(A) > 0$ there exists a constant $c \equiv c_{d,\ell,m,\mu(A)} > 0$ such that*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]})A\right) \geq c. \tag{5}$$

Proof. Write $q_{i,j}(n) = a_{1,i,j}n + \ldots + a_{d,i,j}n^d,\ 1 \le i \le \ell,\ 1 \le j \le m$ (by putting some zero terms if needed). For any $r \in \mathbb{N}$, let

$$S_r = \Big\{ s \in \mathbb{N} :\ \Big\| a_{k,i,j}s^k \Big\| < \frac{1}{2dr^d}\ \ \forall\, 1 \le k \le d,\ 1 \le i \le \ell,\ 1 \le j \le m \Big\}.$$

Use Lemma 3.4 to bound the lower density of S_r below by some constant $c_{d,\ell,m,r} > 0$.

For any $s \in S_r,\ 1 \le n \le r$, using Lemma 3.5, we have

$$[[q_{i,j}(ns)]] = [[a_{1,i,j}s]]n + \ldots + [[a_{d,i,j}s^d]]n^d,\ \ 1 \le i \le \ell,\ 1 \le j \le m.$$

Set

$$p_{s,i,j}(n) = [[a_{1,i,j}s]]n + \ldots + [[a_{d,i,j}s^d]]n^d,\ \ s \in \mathbb{N},\ 1 \le i \le \ell,\ 1 \le j \le m.$$

Use Proposition 3.7 to find a positive constant $\tilde{c} \equiv \tilde{c}_{d,\mu(A),\ell,m}$ and a positive integer $N_0 \equiv N_0(d,\mu(A),\ell,m)$ such that, for some $1 \le n \le N_0$ and any $s \in \mathbb{N}$, to have

$$\mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-p_{s,i,1}(n)})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-p_{s,i,m}(n)})A \right) \ge \tilde{c}.$$

Then,

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]})A \right) \ge$$

$$\frac{1}{N_0}\liminf_{N\to\infty} \frac{1}{N}\sum_{s=1}^{\left[\frac{N}{N_0}\right]}\sum_{n=1}^{N_0} \mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(ns)]]})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(ns)]]})A \right) \ge$$

$$\frac{1}{N_0}\liminf_{N\to\infty} \frac{1}{N}\sum_{s\in S_{N_0}\cap\left\{1,\ldots,\left[\frac{N}{N_0}\right]\right\}}\ \sum_{n=1}^{N_0} \mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-p_{s,i,1}(n)})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-p_{s,i,m}(n)})A \right) \ge$$

$$\frac{\tilde{c}_{d,\mu(A),\ell,m}}{N_0^2}\liminf_{N\to\infty} \frac{\left| S_{N_0} \cap \left\{1,\ldots,\left[\frac{N}{N_0}\right]\right\}\right|}{\left[\frac{N}{N_0}\right]} \ge \frac{\tilde{c}_{d,\mu(A),\ell,m}\cdot c_{d,N_0,\ell,m}}{N_0^2} > 0,$$

from which we have the result. □

By using Proposition 3.8, we immediately obtain the following uniform result, Corollary 3.9, the second ingredient we need in order to prove Theorem 1.2. We note at this point that the uniformity in W is crucial for the proof of Theorem 1.2.

Corollary 3.9. *Let $\ell, m \in \mathbb{N}$, $(X,\mathcal{X},\mu,T_1,\ldots,T_\ell)$ a system and $q_{i,j} \in \mathbb{R}[t]$ polynomials with maximum degree d and $q_{i,j}(0) = 0$ for $1 \le i \le \ell$, $1 \le j \le m$. Then for any $A \in \mathcal{X}$ with $\mu(A) > 0$, there exists a constant $c \equiv c_{d,\ell,m,\mu(A)} > 0$ such that for every $W \in \mathbb{N}$*

$$\liminf_{N\to\infty} \frac{1}{N}\sum_{n=1}^{N} \mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(Wn)]]})A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(Wn)]]})A \right) \ge c. \tag{6}$$

Remark. In fact, it is known that both limits in (5) and (6) exist from [19].

4. Proof of main results

In this section we give the proof of Theorems 1.2, 1.3 and 1.4. First, we prove Theorem 1.2 for the $\mathbb{P}-1$ case (the $\mathbb{P}+1$ case follows similarly).

Proof of Theorem 1.2. Using Proposition 3.2 for the polynomials $q_{i,j}(n-1)+\frac{1}{2}$ with $r=1$ and Corollary 3.9, we have (recall that $[[x]]=[x+1/2]$), for sufficiently large $w\in\mathbb{N}$, that

$$\liminf_{N\to\infty}\frac{1}{N}\sum_{n=1}^{N}\Lambda'_{w,1}(n)\cdot\mu\left(A\cap(\prod_{i=1}^{\ell}T_i^{-[[q_{i,1}(Wn)]]})A\cap\ldots\cap(\prod_{i=1}^{\ell}T_i^{-[[q_{i,m}(Wn)]]})A\right)>0,$$

from which we have the required non-empty intersection with $\mathbb{P}-1$. □

Remark. According to Lemma 2.1, we have that the conclusion of Theorem 1.2, and so of Theorem 1.1, is satisfied for a set of integers n with positive relative density in the shifted primes $\mathbb{P}-1$. The analogous result, by a similar argument, holds for the set $\mathbb{P}+1$ as well.

Proof of Theorem 1.3. We borrow the arguments from the proof of Theorem 1.3 from [13]. By Lemma 2.1 it suffices to show that the following sequence is Cauchy in $L^2(\mu)$:

$$A(N):=\frac{1}{N}\sum_{n=1}^{N}\Lambda'(n)\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(n)]})f_1\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(n)]})f_m.$$

Using Proposition 3.2, for any $\varepsilon>0$, if for $w,r\in\mathbb{N}$, we define

$$B_{w,r}(N):=\frac{1}{N}\sum_{n=1}^{N}(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(Wn+r)]})f_1\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(Wn+r)]})f_m.$$

Then for sufficiently large N and some w_0 (which gives us a corresponding W_0) we have

$$\left\|A(W_0N)-\frac{1}{\phi(W_0)}\sum_{1\leq r\leq W_0,\ (r,W_0)=1}B_{w_0,r}(N)\right\|_{L^2(\mu)}<\varepsilon. \tag{7}$$

Using the fact that for any $1\leq r\leq W_0$ the sequence $(B_{w_0,r}(N))$ is Cauchy in $L^2(\mu)$, which follows from [19], as well as the Relation (7), we get that for M,N sufficiently large

$$\|A(W_0M)-A(W_0N)\|_{L^2(\mu)}<\varepsilon.$$

Then $(A(N))$ is Cauchy in $L^2(\mu)$, since

$$\|A(W_0N+r)-A(W_0N)\|_{L^2(\mu)}=o_N(1),$$

for every $1\leq r\leq W_0$. □

Proof of Theorem 1.4. We use the same argument as in the proof of Theorem 1.2. The only difference is that we need to prove a variant of Corollary 3.9 for the integer part of the polynomials $q_{i,j}(n)=ak_{i,j}n^{d_{i,j}}$, $1\leq i\leq\ell$, $1\leq j\leq m$. Let $d=\max\{d_{i,j}:\ 1\leq i\leq\ell,$ $1\leq j\leq m\}$, $k=\max\{k_{i,j}:\ 1\leq i\leq\ell,1\leq j\leq m\}$, $W\in\mathbb{N}$ and, for $r\in\mathbb{N}$, set

$$S_r=\left\{m\in\mathbb{N}:\ \{aW^{d_{i,j}}m^{d_{i,j}}\}<\frac{1}{kr^d},\ 1\leq i\leq\ell,1\leq j\leq m\right\}$$

(without loss of generality $k \in \mathbb{N}$). It is sufficient to find a positive lower bound for the lower density of S_r, independent of W. Then, since, for any $m \in S_r$ and $1 \leq n \leq r$, we have

$$[q_{i,j}(Wmn)] = [ak_{i,j}W^{d_{i,j}}m^{d_{i,j}}n^{d_{i,j}}] = [ak_{i,j}W^{d_{i,j}}m^{d_{i,j}}]n^{d_{i,j}},$$

we can follow the proof of Proposition 3.8 to obtain the result.

If $a = t/s \in \mathbb{Q}$ with $s \in \mathbb{N}$, then $\underline{d}(S_r) \geq 1/s$.

If $a \notin \mathbb{Q}$, then, if $\{d_{i,j} : \ 1 \leq i \leq \ell, \ 1 \leq j \leq m\} = \{d_1 < \ldots < d_\xi = d\}$, since $(a(nW)^{d_1}, \ldots, a(nW)^{d_\xi})_n$ is equidistributed in $\mathbb{T}^\xi$, we get $d(S_r) = \dfrac{1}{(kr^d)^\xi} \geq \dfrac{1}{(kr^d)^{\ell m}}$. □

5. An application on Gowers uniform sets

In this last section, following [11], we will give an application of our approach, on Gowers uniform sets. More specifically, we will prove Theorem 1.5, i.e., any shift of a Gowers uniform set is a set of closest integer polynomial multiple recurrence and of integer part polynomial multiple mean convergence (see definitions below). The main ingredients in order to obtain this result will be Theorem 5.2 and a result from [11], Lemma 5.3.

Imitating [11], we give the following definitions:

Definition. i) A set of integers S is *a set of integer part polynomial multiple mean convergence* if for every $\ell, m \in \mathbb{N}$, system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$, functions $f_1, \ldots, f_m \in L^\infty(\mu)$ and polynomials $q_{i,j} \in \mathbb{R}[t]$ for $1 \leq i \leq \ell$, $1 \leq j \leq m$, the averages

$$\frac{1}{|S \cap [1,N]|} \sum_{n \in S \cap [1,N]} (\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]}) f_1 \cdot \ldots \cdot (\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]}) f_m$$

converge in $L^2(\mu)$ as $N \to \infty$.

ii) A set of integers S is *a set of closest integer polynomial multiple recurrence* if for every $\ell, m \in \mathbb{N}$, system $(X, \mathcal{X}, \mu, T_1, \ldots, T_\ell)$, set $A \in \mathcal{X}$ with $\mu(A) > 0$ and polynomials $q_{i,j} \in \mathbb{R}[t]$ with $q_{i,j}(0) = 0$, $1 \leq i \leq \ell$, $1 \leq j \leq m$, we have

$$\mu\left(A \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,1}(n)]]}) A \cap \ldots \cap (\prod_{i=1}^{\ell} T_i^{-[[q_{i,m}(n)]]}) A\right) > 0$$

for a set of $n \in S$ with positive lower relative, in S, density.

Remark. The aforementioned definitions are equivalent to the ones in [11] for sets of positive density in $\mathbb{N}$.

The following result is a corollary of Theorems 1.2 and 1.3.

Theorem 5.1. *The set of shifted primes $\mathbb{P} - 1$ (and similarly $\mathbb{P} + 1$) is a set of closest integer polynomial multiple recurrence and the set of primes, $\mathbb{P}$, is a set of integer part polynomial multiple mean convergence.*

We will now prove that $\mathbb{N}$ is a set of closest integer polynomial multiple recurrence and of integer part polynomial multiple mean convergence, which we will use in the proof of Theorem 1.5.

Theorem 5.2. *The set of positive integers, $\mathbb{N}$, is a set of closest integer polynomial multiple recurrence and of integer part polynomial multiple mean convergence.*

Proof. That $\mathbb{N}$ is a set of closest integer polynomial multiple recurrence follows from Proposition 3.8. The integer part polynomial multiple mean convergence of $\mathbb{N}$ follows from Theorem 1.4 of [19]. $\square$

In order to prove Theorem 1.5, we also need the following result of Frantzikinakis and Host, which we borrow from [11]:

Lemma 5.3 ([11, Lemma 2.4])**.** *Let $d \geq 2$ be an integer and $\varepsilon, \kappa > 0$. Then there exist $\eta > 0$ and $N_0 \in \mathbb{N}$, such that for all integers $N, \tilde{N}$ with $N_0 \leq N \leq \tilde{N} \leq \kappa N$, every interval $J \subseteq [1, N]$ and $g : \mathbb{Z}_{\tilde{N}} \to \mathbb{C}$ with $|g| \leq 1$, the following implication holds:*

$$\text{if } \ \|g\|_{U_d(\mathbb{Z}_{\tilde{N}})} \leq \eta, \ \text{ then } \ \|g \cdot \mathbf{1}_J\|_{U_d(\mathbb{Z}_N)} \leq \varepsilon.$$

We will now recall the definition of a Gowers uniform set.

Definition ([11])**.** We say that a set of positive integers S is *Gowers uniform* if there exists a positive constant c such that

$$\lim_{N\to\infty} \|\mathbf{1}_S - c\|_{U_r(\mathbb{Z}_N)} = 0$$

for every $r \in \mathbb{N}$.

Remark. Note that if such a (positive) constant c exists, then, for $r = 1$, it is equal to the density of S, i.e.,

$$c = \lim_{N\to\infty} \frac{|S \cap [1, N]|}{N}.$$

Using a similar argument to the one of Theorem 3.1, we get the following:

Proposition 5.4. *Let $\ell, m \in \mathbb{N}$, $(X, \mathcal{X}, \mu, T_1, \dots, T_\ell)$ a system, $q_{i,j} \in \mathbb{R}[t]$ polynomials, $1 \leq i \leq \ell$, $1 \leq j \leq m$ and $f_1, \dots, f_m \in L^\infty(\mu)$.*

For the sequence of functions

$$b(n) = (\prod_{i=1}^{\ell} T_i^{[q_{i,1}(n)]}) f_1 \cdot \ldots \cdot (\prod_{i=1}^{\ell} T_i^{[q_{i,m}(n)]}) f_m,$$

there exists $d \in \mathbb{N}$, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, such that for every sequence $(a(n))$ with $\|a\|_\infty \leq 1$ and every $0 < \delta < 1$, there exists a constant $C_{d,\delta}$ depending on d and δ, such that

$$\left\| \frac{1}{N} \sum_{n=1}^{N} a(n) \cdot b(n) \right\|_{L^2(\mu)} \leq C_{d,\delta} \left(\|a \cdot \mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})} + o_N(1) \right) + c_\delta,$$

where $c_\delta \to 0$ as $\delta \to 0^+$ and the term $o_N(1)$ depends only on the integer d.

Remark. As in Theorem 3.1, the proof of Proposition 5.4 also depends on the fact that the integer part multicorrelation sequence can be written as a sum of a conditional expectation of a usual multicorrelation sequence and an error function, which we can control.

Proof of Proposition 5.4. Let $0<\delta<1$. For the given transformations on X, we define the same action on $\mathbb{R}^{\ell m}$ as in Theorem 3.1, the function $\hat{f}_0$ and the same Y-extensions $\hat{f}_i$, for $1\leq j\leq m$, in the product probability space $Y=X\times[0,1)^{\ell m}$, endowed with the measure $\nu=\mu\times\lambda^{\ell m}$.

We also define the same sequences of functions $(\tilde{b}(n))$ and $(b'(n))$ in Y and X respectively, as in the proof of Theorem 3.1. Then, by using the triangle and the Cauchy-Schwarz inequality, following the proof of Theorem 3.1, we have that

$$\delta^{\ell m}\left\|\frac{1}{N}\sum_{n=1}^{N}a(n)b(n)\right\|_{L^2(\mu)}\leq\left\|\frac{1}{N}\sum_{n=1}^{N}a(n)\cdot(\delta^{\ell m}b(n)-b'(n))\right\|_{L^2(\mu)}+\left\|\frac{1}{N}\sum_{n=1}^{N}a(n)\tilde{b}(n)\right\|_{L^2(\nu)}.$$

For the second term we can use Lemma 2.3 to find a positive integer $d=d(\ell,m,\max\deg(q_{i,j}))$, such that

$$\left\|\frac{1}{N}\sum_{n=1}^{N}a(n)\tilde{b}(n)\right\|_{L^2(\nu)}\leq C_d\left(\|a\cdot\mathbf{1}_{[1,N]}\|_{U_d(\mathbb{Z}_{dN})}+o_N(1)\right),$$

Where C_d is a constant depending on d. We remark at this point that since the weighted sequence $(a(n))$ is bounded, then the $o_N(1)$ term of Lemma 2.3 only depends on d (see [13]).

Following the proof of Theorem 3.1 we obtain (since $(a(n))$ is bounded by 1), that the first term satisfies

$$\left\|\frac{1}{N}\sum_{n=1}^{N}a(n)\cdot(\delta^{\ell m}b(n)-b'(n))\right\|_{L^2(\mu)}\leq\delta^{\ell m}c_\delta,$$

with $c_\delta\to 0$ as $\delta\to 0^+$, since, independently of $x\in X$, we have that

$$\Big|\frac{1}{N}\sum_{n=1}^{N}(\delta^{\ell m}b(n)(x)-b'(n)(x))\Big|\leq\delta^{\ell m}c_\delta,$$

with $c_\delta\to 0$ as $\delta\to 0^+$. The result now follows. □

We are now ready to prove Theorem 1.5 and close this article. The main ingredients in order to obtain this result is Proposition 5.4, Lemma 5.3, which we borrow from [11], and finally, Theorem 5.2.

Proof of Theorem 1.5. Let $\ell,m\in\mathbb{N}$, $(X,\mathcal{X},\mu,T_1,\ldots,T_\ell)$ system, functions $f_1,\ldots,f_m\in L^\infty(\mu)$ and polynomials $q_{i,j}\in\mathbb{R}[t]$ for $1\leq i\leq\ell$, $1\leq j\leq m$. In order to prove the second conclusion, for any $n\in\mathbb{N}$ let

$$b(n)=(\prod_{i=1}^{\ell}T_i^{[q_{i,1}(n)]})f_1\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{[q_{i,m}(n)]})f_m. \tag{8}$$

If $S\subseteq\mathbb{N}$ is a Gowers uniform set and c is the respective positive constant from the definition, by Proposition 5.4, we have that there exists a positive integer d, depending only on the maximum degree of the polynomials $q_{i,j}$ and the integers ℓ and m, such that for every $0<\delta<1$ there exists a constant $C_{d,\delta}$ depending on d and δ, such that

$$\left\|\frac{1}{N}\sum_{n=1}^{N}(\mathbf{1}_S(n)-c)\cdot b(n)\right\|_{L^2(\mu)}\leq C_{d,\delta}\left(\left\|(\mathbf{1}_S-c)\cdot\mathbf{1}_{[1,N]}\right\|_{U_d(\mathbb{Z}_{dN})}+o_N(1)\right)+c_\delta, \tag{9}$$

where $c_\delta\to 0$ as $\delta\to 0^+$ and the term $o_N(1)$ depends only on the integer d.

From Gowers uniformity, for $r=d$, we have that $\lim_{N\to\infty}\|\mathbf{1}_S-c\|_{U_d(\mathbb{Z}_{dN})}=0$. Then, using Lemma 5.3, for $\kappa=d$, $\tilde{N}=dN$ and $J=[1,N]$, we get

$$\lim_{N\to\infty}\left\|(\mathbf{1}_S-c)\cdot\mathbf{1}_{[1,N]}\right\|_{U_d(\mathbb{Z}_{dN})}=0.$$

So, from (9), we have that

$$\limsup_{N\to\infty}\left\|\frac{1}{N}\sum_{n\in S\cap[1,N]}b(n)-\frac{c}{N}\sum_{n=1}^{N}b(n)\right\|_{L^2(\mu)}\leq c_\delta,$$

hence, by taking $\delta\to 0^+$, we get

$$\lim_{N\to\infty}\left\|\frac{1}{N}\sum_{n\in S\cap[1,N]}b(n)-\frac{c}{N}\sum_{n=1}^{N}b(n)\right\|_{L^2(\mu)}=0. \tag{10}$$

Using the remark following the definition of a Gowers uniform set, c is the density of S, so

$$\lim_{N\to\infty}\frac{|S\cap[1,N]|}{N}=c,$$

hence, by writing

$$\begin{aligned}\frac{1}{N}\sum_{n\in S\cap[1,N]}b(n)-\frac{c}{N}\sum_{n=1}^{N}b(n) &= c\left(\frac{1}{|S\cap[1,N]|}\sum_{n\in S\cap[1,N]}b(n)-\frac{1}{N}\sum_{n=1}^{N}b(n)\right)\\ &+\left(1-c\frac{N}{|S\cap[1,N]|}\right)\frac{1}{N}\sum_{n\in S\cap[1,N]}b(n)\end{aligned}$$

and recalling from the definition of a Gowers uniform set that $c>0$, Relation (10) implies

$$\lim_{N\to\infty}\left\|\frac{1}{|S\cap[1,N]|}\sum_{n\in S\cap[1,N]}b(n)-\frac{1}{N}\sum_{n=1}^{N}b(n)\right\|_{L^2(\mu)}=0. \tag{11}$$

By using (11) and the fact that $\mathbb{N}$ is a set of integer part polynomial multiple mean convergence (from Theorem 5.2), we get the result.

In order to show the first conclusion, for a set $A\in\mathcal{X}$ with $\mu(A)>0$, similarly, we let for every $n\in\mathbb{N}$

$$b(n)=(\prod_{i=1}^{\ell}T_i^{[[q_{i,1}(n)]]})\mathbf{1}_A\cdot\ldots\cdot(\prod_{i=1}^{\ell}T_i^{[[q_{i,m}(n)]]})\mathbf{1}_A.$$

By using (11), the Cauchy-Schwarz inequality and the fact that $\mathbb{N}$ is a set of closest integer polynomial multiple recurrence (from Theorem 5.2), we get the result. □

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