ADDITIVE AND MULTIPLICATIVE RAMSEY THEORY FOR THE SET OF NATURAL NUMBERS

1. Introduction

In these notes we will show some coloring results concerning the additive and multiplicative Ramsey theory via the theory of ultrafilters. We identify every ultrafilter on $\mathbb{N}$ with a finitely additive $\{0,1\}$-valued measure on the power set of $\mathbb{N}$.

The first of the classical results of the Ramsey Theory are due to Hilbert, Schur, and van der Waerden. One may wonder why it is called Ramsey Theory. The reason is that Ramsey's theorem is a more general structural result, not depending on the arithmetic structural of $\mathbb{N}$.

Theorem 1.1 (Ramsey, 1929)**.** *Let M be an infinite subset of $\mathbb{N}$ and $k,\ r \in \mathbb{N}$. If $[M]^k = \{m_1 < \ldots < m_k :\ m_i \in M,\ 1 \le i \le k\}$ and we have $[M]^k = \bigcup_{i=1}^r C_i$, then there exist $1 \le i_0 \le r$ and an infinite subset L of M such that $[L]^k \subseteq C_{i_0}$.*

General form of the results in Ramsey theory: If V is an infinite *highly organized* structure (i.e. a semigroup), then, for any finite partition of $V = V_1 \cup \ldots \cup V_r$, $r \in \mathbb{N}$, there exists $1 \le i_0 \le r$ such that V_{i_0} contains arbitrary large (sometimes even infinite) *highly organized* substructures.

2. Classical coloring results

Firstly we will see Hilbert's lemma which states that whenever we take a finite coloring of natural numbers we can find infinite monochromatic translations of a set of finite sums.

Theorem 2.1 (Hilbert, 1892)**.** *For any finite coloring $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$ and for any $n \in \mathbb{N}$, there exists $1 \le i_0 \le r$ such that for some x_j, $j = 1, \ldots, n$ and for infinitely many t we have*

$$t + FS((x_j)_{j=1}^n) = t + \left\{ \textstyle\sum_{j\in\alpha} x_j :\ \emptyset \ne \alpha \subseteq \{1, \ldots, n\} \right\} \subseteq C_{i_0}.$$

The next classical coloring result of Schur, allows one to omit the translates on the finite sums when $n = 2$.

Theorem 2.2 (Schur, 1916)**.** *If $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$, there exist $1 \le i_0 \le r$ and $x,\ y,\ z \in C_{i_0}$ such that $x + y = z$.*

A more general result of the previous is due to Hindman.

Theorem 2.3 (Hindman, 1974)**.** *Let $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$ a finite coloring of the set of natural numbers. Then, there exist an infinite sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ and $1 \le i_0 \le r$ such that*

$$FS((x_n)_{n\in\mathbb{N}}) = \{\textstyle\sum_{i\in\alpha} x_i :\ \emptyset \ne \alpha \subseteq \mathbb{N},\ |\alpha| < \infty\} \subseteq C_{i_0}.$$

The following result is one of the most (maybe the most) fundamental result of Ramsey Theory. It is due to van der Waerden.

Theorem 2.4 (van der Waerden, 1927)**.** *Whenever the natural numbers are finitely partitioned, one of the cells of the partition contains arbitrary long arithmetic progressions.*

We will see the proofs of the previous theorems in the next sections, using the theory of ultrafilters.

3. Filters and ultrafilters

Definition 3.1. A *filter* p on $\mathbb{N}$ is a set of subsets of $\mathbb{N}$ satisfying the following conditions:
(i) $\emptyset \notin p$,
(ii) if $A \in p$ and $A \subseteq B$, then $B \in p$, and
(iii) if $A \in p$ and $B \in p$, then $A \cap B \in p$.
An *ultrafilter* is a maximal filter.
Let $\beta\mathbb{N}$, the Stone-Čech compactification of $\mathbb{N}$, be the set of ultrafilters on $\mathbb{N}$.

Proposition 3.2. *Every filter p is contained in an ultrafilter q.*

Proof. Let p be a filter on $\mathbb{N}$. We will show that there exists a maximal filter q on $\mathbb{N}$ such that $p \subseteq q$. Let the family

$$\mathcal{C} = \{\mathcal{F} \subseteq \mathcal{P}(\mathbb{N}) :\ \mathcal{F} \text{ filter on } \mathbb{N} \text{ and } p \subseteq \mathcal{F}\},$$

ordered by inclusion. $\mathcal{C} \neq \emptyset$ since $p \in \mathcal{C}$. Let $\mathcal{D} \subseteq \mathcal{C}$ be a non-empty totaly ordered subfamily of $\mathcal{C}$. Then $\mathcal{U} = \bigcup\{\mathcal{F} :\ \mathcal{F} \in \mathcal{D}\}$ is a filter on $\mathbb{N}$ with $\mathcal{U} \in \mathcal{C}$. Using Zorn's lemma we get a maximal element of $\mathcal{C}$, q. q is an ultrafilter on $\mathbb{N}$ with $p \subseteq q$. □

Proposition 3.3. *A filter p on $\mathbb{N}$ is an ultrafilter if and only if for all $A \subseteq \mathbb{N}$, either $A \in p$ or $A^c \in p$.*

Proof. ($\Rightarrow$) Let p an ultrafilter on $\mathbb{N}$ with $A \notin p$. We will show that $A^c \in p$. There exists $B \in p$ with $B \cap A = \emptyset$, for otherwise $p' = \{C \subseteq \mathbb{N} :\ A \cap B \subseteq C \text{ for some } B \in p\}$ is a filter containing p and A, with $A \notin p$. Now we have hat $A^c \in p$, for otherwise we would have that there exists $C \in p$ with $C \cap A^c = \emptyset$, and consequently $\emptyset = B \cap C \in p$.

($\Leftarrow$) Let q be an ultrafilter such that $p \subseteq q$. If $p \neq q$, then, there exists $A \subseteq \mathbb{N}$ such that $A \in q \setminus p$. According to the hypothesis, $A^c \in p \subseteq q$, so, $\emptyset = A \cap A^c \in q$, a contradiction. □

Remark 3.4. Every element $n \in \mathbb{N}$ can be identified with the set $\{A \subseteq \mathbb{N} :\ n \in A\}$ which is an ultrafilter. Such ultrafilters are called *principal.*

Proposition 3.5. *There exist non-principal ultrafilters on $\mathbb{N}$.*

Proof. Take

$$\mathcal{B} = \{A \subseteq \mathbb{N} : \ \mathbb{N} \setminus A \text{ is finite}\}$$

and an ultrafilter p with $\mathcal{B} \subseteq p$. Observe that if A is a finite subset of $\mathbb{N}$, then $\mathbb{N} \setminus A \in \mathcal{B} \subseteq p$, thus, $A \notin p$ and we have the conclusion. □

Remark 3.6. According to Comfort and Negrepontis ([CN]) we cannot produce a non-principal ultrafilter without the use of Zorn's lemma.

Remark 3.7. An alternative way of looking at ultrafilters is to identify each ultrafilter $p \in \beta\mathbb{N}$ with a finitely additive $\{0,1\}$-valued measure μ on the power set of $\mathbb{N}$, $\mathcal{P}(\mathbb{N})$ according to the rule

$$\mu(A) = 1 \iff A \in p.$$

4. Topological structure of $\beta\mathbb{N}$

Definition 4.1. We define a unique topology $\mathcal{T}$ on $\beta\mathbb{N}$ which has as a subbase the family $\mathcal{C} = \{\overline{A} : \ A \subseteq \mathbb{N}\}$, where $\overline{A} = \{\mu \in \beta\mathbb{N} : \ \mu(A) = 1\}$.

Proposition 4.2. *For every* $\emptyset \neq A$, $B \subseteq \mathbb{N}$ *we have:*
(*i*) $\overline{X \setminus A} = \beta\mathbb{N} \setminus \overline{A}$, $\overline{\emptyset} = \emptyset$,
(*ii*) $\overline{A} \cap \overline{B} = \overline{A \cap B}$,
(*iii*) $\overline{A} \cup \overline{B} = \overline{A \cup B}$, *and*
(*iv*) *the family* $\mathcal{C} = \{\overline{A} : \ A \subseteq \mathbb{N}\}$ *is a base for the topology* $\mathcal{T}$.

Proof. (*i*) $\mu(X \setminus A) = 1$ if and only if $\mu(A) = 0$, and $\mu(\emptyset) = 0$ for every $\mu \in \beta\mathbb{N}$.
(*ii*) $\mu(A \cap B) = 1$ if and only if $\mu(A) = 1$ and $\mu(B) = 1$.
(*iii*) $\mu(A \cup B) = 1$ if and only if $\mu(A) = 1$ or $\mu(B) = 1$.
(*iv*) Immediate from (*i*), (*ii*) and (*iii*). □

Theorem 4.3. $\beta\mathbb{N}$ *is a compact Hausdorff space with* $\overline{\mathbb{N}} = \beta\mathbb{N}$.

Proof. Compact : Let a family $(F_i)_{i \in I}$ of closed sets on $\beta\mathbb{N}$ with the finite intersections property. We assume without lose of generality that each $F_i = \overline{A_i}$ for some $A_i \subseteq \mathbb{N}$. Since $\bigcap_{i \in \alpha} \overline{A_i} = \overline{\bigcap_{i \in \alpha} A_i}$ we have that $(A_i)_{i \in I}$ has the finite intersections property as well. Let μ be an ultrafilter containing the filter

$$\mathcal{C} = \{A \subseteq \mathbb{N} : \ A_{i_1} \cap \dots A_{i_r} \subseteq A \text{ for some } i_1, \dots, i_r \in I, \, r \in \mathbb{N}\}.$$

Then $\mu(A_i) = 1$ for every $i \in I$, thus $\mu \in \bigcap_{i\in I} \overline{A_i}$.

Hausdorff : Let $\mu \neq \nu \in \beta\mathbb{N}$. Then, there exists $A \subseteq \mathbb{N}$ such that $\mu(A) = 1$ and $\nu(A) = 0$. Then, $\mu \in \overline{A}$ while $\nu \in \overline{X \setminus A} = \beta\mathbb{N} \setminus \overline{A}$, with $\overline{A} \cap \overline{X \setminus A} = \emptyset$.

Finally, for every every $\emptyset \neq A \subsetneq \mathbb{N}$ and $n_0 \in A$ we have that $\mu_{n_0} \in \overline{A} \cap \{\mu_n : n \in \mathbb{N}\}$, so, $\overline{\mathbb{N}} = \overline{\{\mu_n : n \in \mathbb{N}\}} = \beta\mathbb{N}$. □

Remark 4.4. According to Gillman and Jerison ([GJ]), the cardinality of $\beta\mathbb{N}$ equals that of $\mathcal{P}(\mathcal{P}(\mathbb{N}))$. It follows that $\beta\mathbb{N}$ is not metrizable, as otherwise, being a compact metric space and hence a separable metric space, it would have cardinality not exceeding that of $\mathcal{P}(\mathbb{N})$.

5. Extending addition from $\mathbb{N}$ to $\beta\mathbb{N}$

Since $\overline{\mathbb{N}} = \beta\mathbb{N}$, we will extend the operation of addition from $\mathbb{N}$ to $\beta\mathbb{N}$. Ultrafilters are finite additive measures, so, the extension will take the form of a convolution.

Definition 5.1. For $A \subseteq \mathbb{N}$ we set $A - n = \{m \in \mathbb{N} : m + n \in A\}$ and for $\mu, \nu \in \beta\mathbb{N}$ we define

$$\mu * \nu(A) = \mu(\{n \in \mathbb{N} : \nu(A - n) = 1\}).$$

Remarks 5.2. 1) $\mu * \nu \in \beta\mathbb{N}$, since $\mu * \nu(\mathbb{N}) = 1$, and if $A, B \in \mathbb{N}$ with $A \cap B = \emptyset$ then,

$$\mu * \nu(A \cup B) = \mu(\{n \in \mathbb{N} : \nu(A \cup B - n) = 1\}) = \mu(\{n \in \mathbb{N} : \nu(A - n) + \nu(B - n) = 1\}) = \mu(\{n \in \mathbb{N} : \nu(A - n) = 1\}) + \mu(\{n \in \mathbb{N} : \nu(B - n) = 1\}) = \mu * \nu(A) + \mu * \nu(B),$$

since $(A - n) \cap (B - n) = \emptyset$.

2) The previous definition is analogous to the usual formulas for convolution of measures μ, ν on a locally compact group G:

$$\mu * \nu(A) = \int_G \nu(x^{-1}A)d\mu(x) = \int_G \mu(Ay^{-1})d\nu(x).$$

In case of an abelian semigroup, convolution is commutative. The introduced operation between ultrafilters is non-commutative although $(\mathbb{N}, +)$ is abelian, since ultrafilters as finitely additive measures, they do not obey Fubini's Theorem.

Example that fails commutability. Let $X = \{x_1, x_2, \ldots\}$ and $Y = \{y_1, y_2, \ldots\}$, with $x_i = 2^{2i}$, $y_i = 2^{2i+1}$. If

$$A = \{x_i + y_i : i < j\} \text{ and } B = \{x_i + y_i : i > j\},$$

then, $A \cap B = \emptyset$. Define

$$U = \{C \subseteq \mathbb{N} : X \setminus C \text{ is finite}\},\ V = \{C \subseteq \mathbb{N} : Y \setminus C \text{ is finite}\}$$

and find ultrafilters μ, ν containing U, V respectively. For every x_i the set $A - x_i$ contains a subset $\{y_{i+1}, y_{i+2}, \ldots\}$, so $\nu(A - x_i) = 1$, thus $\mu * \nu(A) = \mu(\{n \in \mathbb{N} : \nu(A - n) = 1) = 1$, since $X \subseteq \{n \in \mathbb{N} : \nu(A - n) = 1\}$, and similarly $\nu * \mu(B) = 1$. $A \cap B = \emptyset$, so, $\mu * \nu \neq \nu * \mu$.

Definition 5.3. A non-empty set X endowed with a topology $\mathcal{T}$ and an operation $+ : X \times X \to X$ is called *right topological semigroup* if

(i) the topological space $(X, \mathcal{T})$ is compact,

(ii) $(X, +)$ is a semigroup, and

(iii) for any $x_0 \in X$ the function $T_{x_0} : X \to X$ with $T_{x_0}(x) = x + x_0$ is continuous.

Theorem 5.4. *$(\beta\mathbb{N}, *, \mathcal{T})$ is a right topological semigroup.*

Proof. For the associativity of the operation $*$, let $A \subseteq \mathbb{N}$ and μ, ν, $\lambda \in \beta\mathbb{N}$. Then, $\mu * (\nu * \lambda)(A) = \mu(\{n \in \mathbb{N} : \ \nu * \lambda(A - n) = 1\}) = \mu(\{n \in \mathbb{N} : \ \nu(\{m \in \mathbb{N} : \lambda((A - n) - m) = 1\}) = 1\}) = \mu(\{n \in \mathbb{N} : \ \nu(\{m \in \mathbb{N} : \ \lambda(A - m) = 1\} - n) = 1\}) = \mu * \nu(\{m \in \mathbb{N} : \ \lambda(A - m) = 1\}) = (\mu * \nu) * \lambda(A)$, since the operation $+$ is associate.

Finally, for $\mu \in \beta\mathbb{N}$ we have that $T : \beta\mathbb{N} \to \beta\mathbb{N}$ with $T_\mu(\nu) = \nu * \mu$ is continuous. Indeed, for $\emptyset \neq A \subseteq \mathbb{N}$ we have that

$$\begin{aligned}(T_\mu)^{-1}(\overline{A}) = \{\nu \in \beta\mathbb{N} : \ \nu * \mu \in \overline{A}\} &= \{\nu \in \beta\mathbb{N} : \ \nu(\{n \in \mathbb{N} : \ \mu(A - n) = 1\}) = 1\} \\ &= \overline{\{n \in \mathbb{N} : \ \mu(A - n) = 1\}}\end{aligned}$$

which is an open set as basic. □

6. Idempotents and IP-sets

Theorem 6.1 (Ellis, 1969)**.** *$\beta\mathbb{N}$ has an idempotent (i.e. there exists an ultrafilter in $\beta\mathbb{N}$ such that $\mu * \mu = \mu$).*

Proof. The proof consists of two steps.

Firstly, we will show that there exists a minimal non-empty right compact subsemigroup of $(\beta\mathbb{N}, *, \mathcal{T})$. Let

$$\mathcal{C} = \{Y \subseteq \beta\mathbb{N} : \ Y \neq \emptyset \text{ and } (Y, *, \mathcal{T}\ |_Y) \text{ is a right compact semigroup}\}.$$

$\mathcal{C} \neq \emptyset$ since $\beta\mathbb{N} \in \mathcal{C}$. Let $\mathcal{D} \subseteq \mathcal{C}$ a totaly ordered subfamily by the rule:

$$A \leq B \iff B \subseteq A \text{ for every } A,\ B \subseteq \beta\mathbb{N}.$$

$\mathcal{D}$ has the finite intersections property and it contains closed subsets of $\beta\mathbb{N}$. Since $\beta\mathbb{N}$ is compact we have that $Z_0 = \bigcap\{Y : \ Y \in \mathcal{D}\}$ is a non-empty compact subset of $\beta\mathbb{N}$. Note that $(Z_0, *, \mathcal{T}\ |_{Z_0})$ is a right topological semigroup and that $Y \leq Z_0$ for any $Y \in \mathcal{D}$. According to Zorn's lemma there exists a non-empty $Y_0 \subseteq \beta\mathbb{N}$, maximal in $\mathcal{C}$, hence $(Y_0, *, \mathcal{T}\ |_{Y_0})$ is a minimal right topological subsemigroup of $(\beta\mathbb{N}, *, \mathcal{T})$.

Secondly, we will show that every element of Y_0 is idempotent. Indeed, we have for $y_0 \in Y_0$ that $Y_0 + y_0 = T_{y_0}(Y_0)$ compact and semigroup, for if $x_1 + y_0$, $x_2 + y_0 \in Y_0 + y_0$, then $(x_1 + y_0) + (x_2 + y_0) = ((x_1 + y_0) + x_2) + y_0 \in Y_0 + y_0$. So, $Y_0 + y_0$ is a right topological semigroup with $Y_0 + y_0 \subseteq Y_0$, so, by minimality, we have that $Y_0 + y_0 = Y_0$. If

$$Y_1 = \{x \in Y_0 : \ x + y_0 = y_0\},$$

then $\emptyset \neq Y_1 = (T_{y_0})^{-1}(\{y_0\})$ is a right compact semigroup (for if $x_1,\ x_2 \in Y_1$ we have that $(x_1 + x_2) + y_0 = x_1 + y_0 = y_0$), with $Y_1 \subseteq Y_0$, thus $y_0 \in Y_0 = Y_1$ idempotent. □

Remark 6.2. Every idempotent in $\beta\mathbb{N}$ is non-principal since $\mu_n * \mu_m = \mu_{n+m}$ for every $n,\ m \in \mathbb{N}$, so, $\mu_n * \mu_n \neq \mu_n$ for every $n \in \mathbb{N}$.

Being a member of an ultrafilter is a notion of largeness since $A \in \mu \ \Leftrightarrow\ \mu(A) = 1$. Being a member of an idempotent is a more special notion of largeness since

$$A \in \mu = \mu * \mu \ \Leftrightarrow\ \mu(\{n \in A :\ \mu(A - n) = 1\}) = 1,$$

which means that for μ-almost all $n \in \mathbb{N}$ the set $A - n$ is μ-big. So, for every n for which $\mu(A-n) = 1$ we have that $\mu(A \cap (A-n)) = 1$, hence we have the analogous of Poincaré's recurrence theorem.

The next result will give us a proof of Hindman's theorem a la Poincaré recurrence.

Theorem 6.3. *Let A be a member of an idempotent in $\beta\mathbb{N}$. Then, there exists an infinite sequence $(x_n)_{n\in\mathbb{N}} \subseteq A$ such that*

$$FS((x_n)_{n\in\mathbb{N}}) = \{\textstyle\sum_{i\in\alpha} x_i :\ \emptyset \neq \alpha \subseteq \mathbb{N},\ |\alpha| < \infty\} \subseteq A.$$

Proof. Let $\mu \in \beta\mathbb{N}$ with $\mu = \mu * \mu$ and $\mu(A) = 1$. Choose $x_1 \in A_1 = A$ such that $\mu(A_1 - x_1) = 1$. Let $A_2 = A_1 \cap (A_1 - x_1)$ and $x_2 \in A_2$ with $\mu(A_2 - x_2) = 1$. Let that we have defined inductively $A_n \subseteq \ldots \subseteq A_1 = A$ and $x_1, \ldots, x_n \in A$ with $x_k \in A_k$, and $\mu(A_k) = \mu(A_k - x_k) = 1$ for every $1 \leq k \leq n$. Set $A_{n+1} = A_n \cap (A_n - x_n)$ and choose $x_{n+1} \in A_{n+1}$ such that $\mu(A_{n+1} - x_{n+1}) = 1$.

We will show with induction on k that $x_{n_1} + \ldots + x_{n_k} \in A_{n_1} \subseteq A_1$ for every $k \in \mathbb{N}$ and $n_1 < \ldots < n_k \in \mathbb{N}$. $x_{n_1} \in A_{n_1}$, so, the claim holds for $k = 1$. Let $n_1 < \ldots < n_{k+1} \in \mathbb{N}$. According to induction hypothesis we have that $x_{n_2} + \ldots + x_{n_{k+1}} \in A_{n_2} \subseteq A_{n_1+1} = A_{n_1} \cap (A_{n_1} - x_{n_1})$, hence

$$x_{n_1} + x_{n_2} + \ldots + x_{n_{k+1}} = x_{n_1} + (x_{n_2} + \ldots + x_{n_{k+1}}) \in A_{n_1}$$

and we have the conclusion. □

An immediate corollary of the previous result is Hindman's Theorem.

Theorem 6.4 (Hindman, 1974)**.** *Let $\mathbb{N} = \bigcup_{i=1}^{r} C_i$, $r \in \mathbb{N}$ a finite coloring of the set of natural numbers. Then, there exist an infinite sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ and $1 \leq i_0 \leq r$ such that*

$$FS((x_n)_{n\in\mathbb{N}}) \subseteq C_{i_0}.$$

Proof. Let $\mu \in \beta\mathbb{N}$ idempotent. Then, there exists $1 \leq i_0 \leq r$ such that $\mu(C_{i_0}) = 1$. □

The notion of largeness which is behind Hindman's theorem is that of being a member of an idempotent ultrafilter. The IP-set $FS((x_n)_{n\in\mathbb{N}})$ that was found inside an μ-big set it is not necessarily itself μ-big. We will now show that any IP-set it is μ-big for some μ idempotent, so, we can say that Hindman's theorem is the density version of itself.

Theorem 6.5. *For any sequence $(x_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ there exists an idempotent $\mu \in \beta\mathbb{N}$ such that $\mu(FS((x_n)_{n\in\mathbb{N}})) = 1$.*

Proof. Let $\Gamma = \bigcap_{m\in\mathbb{N}} \overline{FS((x_n)_{n\geq m})} \subseteq \beta\mathbb{N}$. We have that Γ is a non-empty compact subset of $\beta\mathbb{N}$ since it has the finite intersections property. We will show that Γ is a semigroup.

Let $\mu,\ \nu \in \Gamma$ we will show that $\mu * \nu \in \Gamma$, equivalently, for $m_0 \in \mathbb{N}$ arbitrary, we show that $\mu*\nu \in \overline{FS((x_n)_{n\geq m_0})}$. Let $A \subseteq \mathbb{N}$ with $\mu*\nu \in \overline{A}$, we will find $n \in FS((x_n)_{n\geq m_0})\cap A$.

If $B = \{n \in \mathbb{N} :\ \nu(A - n) = 1\}$, we have that $1 = \mu * \nu(A) = \mu(B) = 1$, so, $\mu \in \overline{B}$, hence there exists $n_1 = \sum_{i\in F} x_i \in FS((x_n)_{n\geq m_0})$ with $F \subseteq \{m_0, \dots, m_1\}$, with $\nu(A - n_1) = 1$. Since $\nu \in \overline{FS((x_n)_{n\geq m_1})}$, we can take $n_2 \in FS((x_n)_{n\geq m_1}) \cap (A - n_1)$. We now have that $n_1 + n_2 \in FS((x_n)_{n\geq m_0}) \cap A$. As a compact right topological semigroup, Γ has an idempotent λ, hence $\lambda(FS((x_n)_{n\in\mathbb{N}})) = 1$. ☐

We showed that a set $A \subseteq \mathbb{N}$ is a member of an idempotent if and only if it contains an IP-set. Since there exist IP-sets that do not even contain arithmetic progressions of length 3 (take for example the set $FS((5^n)_{n\in\mathbb{N}})$), not every idempotent may reveal something about van der Waerden's theorem.

7. Minimal left ideals, minimal idempotents and central sets

Definition 7.1. Let $\emptyset \neq I \subseteq \beta\mathbb{N}$. I is a *left ideal* of $\beta\mathbb{N}$ if

$$\beta\mathbb{N} + I = \{\mu * \nu :\ \mu \in \beta\mathbb{N},\ \nu \in I\} \subseteq I.$$

I is a *right ideal* of $\beta\mathbb{N}$ if

$$I + \beta\mathbb{N} = \{\nu * \mu :\ \nu \in I,\ \mu \in \beta\mathbb{N}\} \subseteq I.$$

I is a *two-sited ideal* of $\beta\mathbb{N}$ if it is simultaneously a left and a right ideal of $\beta\mathbb{N}$.

Remarks 7.2. (1) For any $\mu \in \beta\mathbb{N}$, $I = \beta\mathbb{N} + \mu$ is a compact left ideal of $\beta\mathbb{N}$.

Indeed, if $\kappa \in \beta\mathbb{N}$ and $\nu = \lambda * \mu \in I$ for some $\lambda \in \beta\mathbb{N}$, then $\kappa * \nu = \kappa * (\lambda * \mu) = (\kappa * \lambda) * \mu \in \beta\mathbb{N} + \mu$. Also, $I = T_\mu(\beta\mathbb{N})$ and we have the compactness.

(2) For every I left ideal of $\beta\mathbb{N}$ and $\mu \in \beta\mathbb{N}$, we have that $I + \mu$ is a left ideal of $\beta\mathbb{N}$.

Indeed, $\beta\mathbb{N} + (I + \mu) = (\beta\mathbb{N} + I) + \mu \subseteq I + \mu$, since I is a left ideal of $\beta\mathbb{N}$.

Proposition 7.3. *Let I be a left ideal of $\beta\mathbb{N}$. Then, there exists a minimal left ideal I_1 of $\beta\mathbb{N}$ with $I_1 \subseteq I$.*

Proof. Let

$$\mathcal{C} = \{J \subseteq \beta\mathbb{N} :\ J \text{ non-empty compact left ideal of } \beta\mathbb{N} \text{ with } J \subseteq I\}.$$

$\mathcal{C} \neq \emptyset$ since for $\mu \in I$, $\beta\mathbb{N} + \mu = T_\mu(\beta\mathbb{N})$ compact left ideal with $\beta\mathbb{N} + \mu \subseteq \beta\mathbb{N} + I \subseteq I$. If $\mathcal{D} = \{J_i\}_i$ is a totaly ordered subfamily of $\mathcal{C}$ we have that $\mathcal{D}$ has the finite intersections property, so, if $J_0 = \bigcap_i Ji$, we have that $J_0 \neq \emptyset$, compact and that $J_0 \in \mathcal{C}$. Indeed, $J_0 \subseteq \beta\mathbb{N}$ and for $\nu \in J_0$ we have $\mu * \nu \in J_i$ for every i, $\mu \in \beta\mathbb{N}$, so, $\mu * \nu \in J_0$, thus $\beta\mathbb{N} + J_0 \subseteq J_0 \subseteq I$. According to Zorn's lemma, $\mathcal{C}$ contains a minimal element, I_1. □

Remarks 7.4. For any minimal left ideal I of $\beta\mathbb{N}$ we have $I = I + \mu = \beta\mathbb{N} + \mu$ for any $\mu \in I$.

Indeed, for $\mu \in I$ we have that $I + \mu \subseteq \beta\mathbb{N} + \mu \subseteq \beta\mathbb{N} + I \subseteq I$. Since $I + \mu$ is a left ideal of $\beta\mathbb{N}$ we have equalities from the minimality of I.

Proposition 7.5. *Let I be a minimal left ideal of $\beta\mathbb{N}$. Then, $I + \mu$ is a minimal left ideal of $\beta\mathbb{N}$ for every $\mu \in \beta\mathbb{N}$.*

Proof. Let $\mu \in \beta\mathbb{N}$ and $I_1 \subseteq I + \mu$ with I_1 left ideal of $\beta\mathbb{N}$. Let

$$A = \{\nu \in I : \ \nu * \mu \in I_1\} \subseteq I.$$

Firstly we will show that A is a left ideal, so, by the minimality of I we will have that $A = I$. For $\nu \in \beta\mathbb{N}$, $\lambda \in A$ we have that $\nu * \lambda \in \beta\mathbb{N} + A \subseteq \beta\mathbb{N} + I \subseteq I$ and $(\nu * \lambda) * \mu = \nu * (\lambda * \mu) \in \beta\mathbb{N} + I_1 \subseteq I_1$, so, $\nu * \lambda \in A$. Finally, we will show that $A + \mu = I_1$. Indeed, $I_1 \subseteq I + \mu = A + \mu$. Conversely, for $\lambda \in A \subseteq I$ we have that $\lambda * \mu \in I_1$, so $A + \mu \subseteq I_1$. Now, we have that $I_1 = A + \mu = I + \mu$, hence, $I + \mu$ is minimal. □

Definition 7.6. For μ, $\nu \in \beta\mathbb{N}$ idempotents we define the order:

$$\mu \leq \nu \iff \mu * \nu = \nu * \mu = \mu.$$

Proposition 7.7. *Let I be a minimal left ideal of $\beta\mathbb{N}$. Then, every idempotent element of I is minimal according to the order $\leq$.*

Proof. Since I is a compact right topological semigroup, there exists an idempotent $\mu \in I$. Let $\nu \in \beta\mathbb{N}$ idempotent with $\nu \leq \mu$. Then, $\nu = \nu * \mu \in \beta\mathbb{N} + I \subseteq I$, so, from the minimality of I we have that $I + \nu = I$. $\mu \in I$, so there exists $\lambda \in I$ such that $\mu = \lambda * \nu$, so, $\nu = \mu * \nu = (\lambda * \nu) * \nu = \lambda * (\nu * \nu) = \lambda * \nu = \mu$. □

Proposition 7.8. *Let μ an idempotent element of $\beta\mathbb{N}$. μ is minimal if and only if μ is an element of a minimal left ideal of $\beta\mathbb{N}$.*

Proof. ($\Rightarrow$) Let μ a minimal idempotent and I a minimal left ideal of $\beta\mathbb{N}$. We have that $I+\mu$ is a minimal compact left ideal, so, it contains an idempotent, ν. Write $\nu = \lambda*\mu$, with $\lambda \in I$, and set $p = \mu*\lambda*\mu \in \mu+I+\mu \subseteq I+\mu$. Since $\mu*\mu = \mu$ we have that $p*\mu = \mu*p = p$, and since $p*p = \mu*\lambda*(\mu*\mu)*\lambda*\mu = \mu*(\lambda*\mu)*(\lambda*\mu) = \mu*(\nu*\nu) = \mu*\nu = p$, we have that $p \leq \mu$, with $p \in I+\mu$.

($\Leftarrow$) Immediate from the previous proposition. □

Proposition 7.9. *Let J be a two-sited ideal of $\beta\mathbb{N}$. Then, for every minimal left ideal I of $\beta\mathbb{N}$ we have that $I \subseteq J$.*

Proof. Note that $J+I \subseteq J+\beta\mathbb{N} \subseteq J$ and $J+I \subseteq \beta\mathbb{N}+I \subseteq I$, hence $J \cap I \neq \emptyset$. So, we have that $J \cap I$ is a left ideal with $J \cap I \subseteq I$, so, by the minimality of I, $I = J \cap I \subseteq J$. □

Definition 7.10. A subset $A \subseteq \mathbb{N}$ is called *central* if there exists a minimal idempotent $\mu \in \beta\mathbb{N}$ with $\mu(A) = 1$.

Proposition 7.11. *For every $n_0 \in \mathbb{N}$ the function $S_{n_0} : \beta\mathbb{N} \to \beta\mathbb{N}$ with $S_{n_0}(\mu) = \mu_{n_0} * \mu$ is continuous.*

Proof. For every $A \subseteq \mathbb{N}$ we have that $(S_{n_0})^{-1}(\overline{A}) = \{\mu \in \beta\mathbb{N} : \ \mu_{n_0} * \mu(A) = 1\} = \{\mu \in \beta\mathbb{N} : \ \mu_{n_0}(\{n \in \mathbb{N} : \ \mu(A-n) = 1\}) = 1\} = \{\mu \in \beta\mathbb{N} : \ \mu(A-n_0) = 1\} = \overline{A-n_0}$. □

Lemma 7.12. *Let $G = (\beta\mathbb{N})^k$, $k \in \mathbb{N}$ with respect to product topology and coordinatewise $*$. Define*

$E_0 = \big\{(a, a+d, \ldots, a+(k-1)d) : \ a \in \mathbb{N}, \ d \in \mathbb{N} \cup \{0\}\big\}$, *and*

$J_0 = \big\{(a, a+d, \ldots, a+(k-1)d) : \ a, \ d \in \mathbb{N}\big\}$.

Let $E = cl_G E_0$ and $J = cl_G J_0$ the closures in G. Then, E is a compact semigroup in G and J is a two-sited ideal of E.

Proof. We show that if $\widetilde{\mu} = (\mu, \ldots, \mu)$ and $\widetilde{\nu} = (\nu, \ldots, \nu)$ belong to E, then $\widetilde{\mu} * \widetilde{\nu} \in E$ and if either $\widetilde{\mu}$ or $\widetilde{\nu}$ is in J, then $\widetilde{\mu} * \widetilde{\nu} \in J$.

Let $V_1 \times \ldots \times V_k \subseteq \mathbb{N}^k$ with $\widetilde{\mu} * \widetilde{\nu}(V_1 \times \ldots \times V_k) = 1$. By right continuity pick a neighborhood of $\widetilde{\nu}$, $\overline{U_1} \times \ldots \times \overline{U_k}$, such that $\overline{U_1} \times \ldots \times \overline{U_k} + \overline{\mu} \subseteq \overline{V_1} \times \ldots \times \overline{V_k}$. Pick $a \in \mathbb{N}$ and $d \in \mathbb{N} \cup \{0\}$ ($d \in \mathbb{N}$ if $\widetilde{\nu} \in J$) such that $\widetilde{x} = (a, a+d, \ldots, a+(k-1)d) \in U_1 \times \ldots \times U_k$. Then, $S_{\mu_{\widetilde{x}}}(\widetilde{\mu}) = \mu_{\widetilde{x}} * \widetilde{\mu} \in \overline{V_1} \times \ldots \times \overline{V_k}$. From the continuity of $S_{\mu_{\widetilde{x}}}$ pick a neighborhood $\overline{W_1} \times \ldots \times \overline{W_k}$ of $\widetilde{\mu}$ with $\mu_{\widetilde{x}} + \overline{W_1} \times \ldots \times \overline{W_k} \subseteq \overline{V_1} \times \ldots \times \overline{V_k}$. Pick $b \in \mathbb{N}$ and $c \in \mathbb{N} \cup \{0\}$ ($c \in \mathbb{N}$ if $\widetilde{\mu} \in J$) such that $\widetilde{y} = (b, b+c, \ldots, b+(k-1)c) \in W_1 \times \ldots \times W_k$. Then, we have that $\mu_{\widetilde{x}} * \mu_{\widetilde{y}} \in \overline{V_1} \times \ldots \times \overline{V_k}$, with $\widetilde{x} + \widetilde{y} = \big(a+b, (a+b)+(d+c), \ldots, (a+b)+(k-1)(d+c)\big)$, while $d+c > 0$ either $\widetilde{\nu} \in J$ or $\widetilde{\mu} \in J$. □

Theorem 7.13. *Every central set $A \subseteq \mathbb{N}$ contains arbitrary long arithmetic progressions.*

Proof. Let $k \in \mathbb{N}$. We will show that A contains an arithmetic progressions of length k. Set $G = (\beta\mathbb{N})^k$. Then, G is a compact right topological semigroup with respect to product topology and coordinatewise $*$. Let

$$E_0 = \{(a, a+d, \ldots, a+(k-1)d) : \ a \in \mathbb{N}, \ d \in \mathbb{N} \cup \{0\}\}, \text{ and}$$
$$J_0 = \{(a, a+d, \ldots, a+(k-1)d) : \ a, \ d \in \mathbb{N}\}.$$

Then, E_0 is a semigroup in $\mathbb{N}^k$ and J_0 is a two-sited ideal of E_0. Let $E = cl_G E_0$ and $J = cl_G J_0$ the closures in G. Then, we have that E is a compact semigroup in G and J is a two-sited ideal of E. Let $\mu \in \beta\mathbb{N}$ a minimal idempotent and set $\widetilde{\mu} = (\mu, \ldots, \mu) \in G$.

Claim $\widetilde{\mu} \in J$.

Let $\overline{A_1} \times \ldots \times \overline{A_k}$ a neighborhood of $\widetilde{\mu}$. Then $\overline{A_1 \cap \ldots \cap A_k}$ is a neighborhood of μ. Then For $a \in \bigcap_{i=1}^k A_i$ we have that $(\mu_a, \ldots, \mu_a) \in (\overline{A_1} \times \ldots \times \overline{A_k}) \cap E$, so, $\widetilde{\mu} \in E$. μ is minimal, so, there exists a minimal left ideal I of $\beta\mathbb{N}$ such that $\mu \in I$. $\widetilde{\mu} \in E$, $E + \widetilde{\mu}$ is a left ideal of E (indeed $E + (E + \widetilde{\mu}) = (E + E) + \widetilde{\mu} \subseteq E + \widetilde{\mu}$), so, there exists a minimal left ideal $\widetilde{I}$ of E such that $\widetilde{I} \subseteq E + \widetilde{\mu}$. Let $\widetilde{\nu} = (\nu_1, \ldots, \nu_k)$ an idempotent in $\widetilde{I}$, so, for some $\widetilde{\lambda} = (\lambda_1, \ldots, \lambda_k) \in E$ we have $\nu_i = \lambda_i * \mu \in \beta\mathbb{N} + I \subseteq I$, hence $I = \beta\mathbb{N} + \nu_i$ for every $i = 1, \ldots, k$. We now have that $\mu \in \beta\mathbb{N} + \nu_i$ for each $i = 1, \ldots, k$, so, we can find $t_i \in \beta\mathbb{N}$ such that $\mu = t_i * \nu_i$, $i = 1, \ldots, k$. Then, $\mu * \nu_i = t_i * (\nu_i * \nu_i) = t_i * \nu_i = \mu$, so, $\widetilde{\mu} = \widetilde{\mu} * \widetilde{\nu} \in E + \widetilde{\nu} \subseteq E + \widetilde{I} \subseteq \widetilde{I}$. So, $\widetilde{\mu}$ is a minimal idempotent, which implies that $\widetilde{\mu} \in J$ from the previous proposition.

To finish the proof, let $A \subseteq \mathbb{N}$ such that $\mu(A) = 1$. Then $\widetilde{\mu} \in \overline{A} \times \ldots \times \overline{A} = (\overline{A})^k$, so, $\widetilde{\mu} \in (\overline{A})^k \cap J = cl_G(A^k \cap J_0)$, hence, $A^k \cap J_0 \neq \emptyset$. It follows that there exist some $a, d \in \mathbb{N}$ with $\{a, \ldots, a+(k-1)d\} \subseteq A$. □

Theorem 7.14 (van der Waerden, 1927)**.** *Whenever the natural numbers are finitely partitioned, one of the cells of the partition contains arbitrary long arithmetic progressions.*

Proof. Let $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$, be a finite coloring of $\mathbb{N}$. Pick $\mu \in \beta\mathbb{N}$ a minimal idempotent. Then, there exists $1 \leq i_0 \leq r$, such that $\mu(C_{i_0}) = 1$, so, C_{i_0} is central. □

8. Extending multiplication from $\mathbb{N}$ to $\beta\mathbb{N}$

We can analogously extend the operation of multiplication from $\mathbb{N}$ to $\beta\mathbb{N}$. Defining for $A \subseteq \mathbb{N}$, $A/n = \{m \in \mathbb{N} : \ m \cdot n \in A\}$, for μ, $\nu \in \beta\mathbb{N}$ we set

$$\mu \circ \nu(A) = \mu(\{n \in \mathbb{N} : \ \nu(A/n) = 1\}).$$

Using this extension, we can have the multiplicative versions of all the previous results.

For example the multiplicative versions of Hindman's and van der Waerden's theorem respectively, are the following.

Theorem 8.1. *Let* $\mathbb{N} = \bigcup_{i=1}^r C_i$, $r \in \mathbb{N}$ *a finite coloring of the set of natural numbers. Then, there exist an infinite sequence* $(x_n)_{n \in \mathbb{N}} \subseteq \mathbb{N}$ *and* $1 \leq i_0 \leq r$ *such that*

$$FP((x_n)_{n\in\mathbb{N}}) = \{\prod_{i\in\alpha} x_i : \emptyset \neq \alpha \subseteq \mathbb{N},\ |\alpha| < \infty\} \subseteq C_{i_0}.$$

Theorem 8.2. *Whenever the natural numbers are finitely partitioned, one of the cells of the partition contains arbitrary long geometric progressions.*

So, whenever we have a finite coloring of natural numbers, $\mathbb{N} = \bigcup_{i=1}^{r} C_i,\ r \in \mathbb{N}$, we can have the previous additive and multiplicative results for some sets C_{i_0} and C_{j_0} respectively. According to Bergelson and Hindman, we can have $i_0 = j_0$.

9. Both additively and multiplicatively central sets

In the set of ultrafilters $\beta\mathbb{N}$ we have in general that the distributive law fails. There exist $\mu, \nu, \lambda \in \beta\mathbb{N}$ such that $(\mu * \nu) \circ \lambda \neq (\mu \circ \lambda) * (\nu \circ \lambda)$ (this in not trivial at all, for an example see [HS]). A special case thought, is true.

Lemma 9.1. *Let $\mu, \nu \in \beta\mathbb{N}$. Then, for every $x \in \mathbb{N}$ we have*

$$(\mu * \nu) \circ \mu_x = (\mu \circ \mu_x) * (\nu \circ \mu_x).$$

Proof. For every $A \subseteq \mathbb{N}$, we have that

$$(\mu * \nu) \circ \mu_x(A) = (\mu * \nu)(A/x) = \mu(\{n \in \mathbb{N} :\ \nu(A/x - n) = 1\}),$$

and

$$(\mu \circ \mu_x) * (\nu \circ \mu_x)(A) = (\mu \circ \mu_x)(\{n \in \mathbb{N} :\ \nu(\{s \in \mathbb{N} :\ \mu_x(\{\lambda \in \mathbb{N} :\ s \cdot \lambda \in \{m \in \mathbb{N} : n + m \in A\}\}) = 1\}) = 1\}) = \mu(\{n \in \mathbb{N} :\ \nu((A - n)/x) = 1\}/x).$$

So, we have to show that if

$$I = \{n \in \mathbb{N} :\ \nu(A/x - n) = 1\} \text{ and } J = \{m \in \mathbb{N} :\ \nu((A - m)/x) = 1\}/x,$$

then $I = J$. $J = \{n \in \mathbb{N} :\ n \cdot x \in \{m \in \mathbb{N} :\ \nu((A - m)/x) = 1\}\} = \{n \in \mathbb{N} : \nu((A - n \cdot x)/x) = 1\}\}$. Since $(A - n \cdot x)/x = \{m \in \mathbb{N} :\ m \cdot x \in A - n \cdot x\} = \{m \in \mathbb{N} : m \cdot x + n \cdot x \in A\} = \{m \in \mathbb{N} :\ (m + n) \cdot x \in A\} = A/x - n$, we have that $J = I$. ☐

The following lemma provides us with a characterization of the minimal left ideals of the form $\beta\mathbb{N} * \mu$ for some $\mu \in \beta\mathbb{N}$.

Lemma 9.2. *Let $\mu \in \beta\mathbb{N}$. Then $\beta\mathbb{N} + \mu$ is a minimal left ideal of $\beta\mathbb{N}$ if and only if given any $A \subseteq \mathbb{N}$, if there is some $n \in \mathbb{N}$ with $\mu(A - n) = 1$, then there is some finite $F \subseteq \mathbb{N}$ such that for all $m \in \mathbb{N}$, $\mu((\bigcup_{n\in F}(A - n)) - m) = 1$.*

Proof. ($\Rightarrow$) Let $C(\mu) = \{A \subseteq \mathbb{N} :\ \mu(A - n) = 1\ \forall\ n \in \mathbb{N}\}$. If $A \subseteq \mathbb{N}$, $\mu \in \beta\mathbb{N}$ and $\mathbb{N} \setminus A \notin C(\mu)$, then $C(\mu) \cup \{A\}$ has the finite intersection property. Indeed, since $\mathbb{N} \in C(\mu)$, $\emptyset \notin C(\mu)$, $C(\mu)$ is closed under hypersets and if A, $B \in C(\mu)$ then

$$\{n \in \mathbb{N} :\ \mu((A \cap B) - n) = 0\} = \{n \in \mathbb{N} :\ \mu(A - n) = 0\} \cup \{n \in \mathbb{N} :\ \mu(B - n) = 0\},$$

we have that $A\cap B\in C(\mu)$, thus $C(\mu)$ is a filter. So, it is sufficient to show that $D\cap A\neq\emptyset$ for $D\in C(\mu)$.

$\mathbb{N}\setminus A\notin C(\mu)$, so $\{n\in\mathbb{N}:\ \mu(A-n)=1\}\neq\emptyset$ and since $D\in C(\mu)$, $\{n\in\mathbb{N}:\ \mu(D-n)=0\}=\emptyset$, thus $|\{n\in\mathbb{N}:\ \mu((A\cap D)-n)=1\}|\geq 1$, so $A\cap D\neq\emptyset$.

Pick $r\in\beta\mathbb{N}$ such that $C(\mu)\cup\{A\}\subseteq r$.

Let that for every F finite subset of $\mathbb{N}$, we have $\bigcup_{x\in F}(A-x)\notin C(\mu)$. Then, since $\{\mathbb{N}\setminus\bigcup_{x\in F}(A-x):\ F$ finite subset of $\mathbb{N}\}$ is closed under finite intersections, we have using the previous argument that $C(\mu)\cup\{\mathbb{N}\setminus\bigcup_{x\in F}(A-x)\}$ has the finite intersection property for every F finite subset of $\mathbb{N}$.

Let $\nu\in\beta\mathbb{N}$ such that $C(\mu)\cup\{\mathbb{N}\setminus\bigcup_{x\in F}(A-x):\ F$ finite subset of $\mathbb{N}\}\subseteq\nu$.

Claim $\{\lambda\in\beta\mathbb{N}:\ C(\mu)\subseteq\lambda\}=\beta\mathbb{N}+\mu$.

($\supseteq$) Let $\lambda\in\beta\mathbb{N}+\mu$. Then $\lambda=r*\mu$, $r\in\beta\mathbb{N}$. Let $A\in C(\mu)$, then

$$\lambda(A)=(r*\mu)(A)=r(\{n\in\mathbb{N}:\ \mu(A-n)=1\})=1.$$

($\subseteq$) Let $\lambda\in\beta\mathbb{N}:\ C(\mu)\subseteq\lambda$. If $\lambda(A)=1$ we have that $\mathbb{N}\setminus A\notin C(\mu)$, so, since $\{\{n\in\mathbb{N}:\ \mu(A-n)=1\}:\ \lambda(A)=1\}$ has the finite intersections property, pick $r\in\beta\mathbb{N}$ such that $\{\{n\in\mathbb{N}:\ \mu(A-n)=1\}:\ \lambda(A)=1\}\subseteq r$. Then, $\lambda\subseteq r*\mu$ and since λ, $r*\mu$ are ultrafilters, we have that $\lambda=r*\mu$.

$\beta\mathbb{N}+\mu$ is minimal and according to the claim, contains the ultrafilters r, ν. $\beta\mathbb{N}+\nu$ is a left ideal and $\beta\mathbb{N}+\nu\subseteq\beta\mathbb{N}+(\beta\mathbb{N}+\mu)\subseteq\beta\mathbb{N}+\mu$, so, from minimality we have that $r\in\beta\mathbb{N}+\mu=\beta\mathbb{N}+\nu$. Let $s\in\beta\mathbb{N}:r=s*\nu$. $r(A)=1$, so, $s(\{x\in\mathbb{N}:\ \nu(A-n)=1\})=1$. Pick $n\in\mathbb{N}$ such that $\nu(A-n)=1$. Since $\{n\}$ is a finite subset of $\mathbb{N}$, we have as well that $\nu(\mathbb{N}\setminus(A-n))=1$, a contradiction.

($\Leftarrow$) It suffices to let ν, $r\in\beta\mathbb{N}+\mu$ and show that there is some $s\in\beta\mathbb{N}:\ r=s*\nu$ (for suppose we have a left ideal I of $\beta\mathbb{N}$ with $I\subsetneq\beta\mathbb{N}+\mu$. If $\nu\in I$, $r\in(\beta\mathbb{N}+\mu)\setminus I$, then $s*\nu\in\beta\mathbb{N}+I\subseteq I$, so, $r\neq s*\nu$).

$C(\mu)\subseteq\nu$ and $C(\mu)\subseteq r$. As in the previous claim, it suffices to show that for every $A\subseteq\mathbb{N}$ with $r(A)=1$, that $\{n\in\mathbb{N}:\ \nu(A-n)=1\}\neq\emptyset$. Let $A\subseteq\mathbb{N}$ with $r(A)=1$. $C(\mu)\subseteq r$, so $\mathbb{N}\setminus A\notin C(\mu)$. Pick from the hypothesis, a finite subset of $\mathbb{N}$, F_0, such that $\bigcup_{x\in F_0}(A-x)\in C(\mu)\subseteq\nu$. Then, $\{n\in\mathbb{N}:\ \nu(A-n)=1\}\neq\emptyset$. □

In order to show the existence of a set which is both additively and multiplicatively central we need the following result which shows the interaction between the operations $*$ and $\circ$.

Theorem 9.3. *Let $\mathcal{M}=cl\{\mu:\ \mu$ is a minimal idempotent in $(\beta\mathbb{N},*)\}$. Then, $\mathcal{M}$ is a left ideal in $(\beta\mathbb{N},\circ)$.*

Proof. Let $\mu\in\mathcal{M}$, $r\in\beta\mathbb{N}$ and let $A\in r\circ\mu\in\beta\mathbb{N}\cdot\mathcal{M}$, so, $(r\circ\mu)(A)=1$. We will show that there exists some minimal idempotent of $(\beta\mathbb{N},*)$ in $\overline{A}$.

$r(\{x\in\mathbb{N}:\ \mu(A/x)=1\})=1$, so, pick $x\in\mathbb{N}$ such that $\mu(A/x)=1$. $\mu\in\mathcal{M}$, so there exists a minimal idempotent ν of $(\beta\mathbb{N},*)$ such that $\nu(A/x)=1$. Thus,

$$(\nu\circ\mu_x)(A)=\nu(A/x)=1.$$

According to Lemma 9.1 we have that $(\nu \circ \mu_x) * (\nu \circ \mu_x) = ((\nu * \nu) \circ \mu_x) = (\nu \circ \mu_x)$. So, $\nu \circ \mu_x$ is an idempotent with $(\nu \circ \mu_x)(A) = 1$. We will show that it is minimal. Since $\nu \circ \mu_x = (\nu \circ \mu_x) * (\nu \circ \mu_x) \in \beta\mathbb{N} + (\nu \circ \mu_x)$, it is sufficient to show that $\beta\mathbb{N} + (\nu \circ \mu_x)$ is a minimal left ideal. ν is in some minimal left ideal R and $\beta\mathbb{N} + \nu \subseteq \beta\mathbb{N} + R \subseteq R$, so $\beta\mathbb{N} + \nu = R$.

Let $B \subseteq \mathbb{N}$ and assume that there is some $n \in \mathbb{N}$ such that $(\nu \circ \mu_x)(B - n) = 1$. We will produce a finite $F \subseteq \mathbb{N}$ such that for all $y \in \mathbb{N}$, $(\nu \circ \mu_x)((\bigcup_{z\in F}(B - z)) - y) = 1$ and we will have the result from Lemma 9.2.

If $n \in \mathbb{N}$ with $(\nu \circ \mu_x)(B - n) = 1$, then

$$1 = \nu(\{s \in \mathbb{N} :\ \mu_x(\{m \in \mathbb{N} :\ s \cdot m \in B - n\}) = 1\}) = \nu(\{s \in \mathbb{N} :\ s \cdot x + n \in B\}).$$

Pick $i \in \{0, 1, \ldots, x - 1\}$ such that $n + i = \lambda \cdot x$. Then,

$$(\nu \circ \mu_x)((B + i) - (n + i)) = 1,$$

since $(B+i)-(n+i) = \{m \in \mathbb{N} :\ m+(n+i) \in B+i\} = \{m \in \mathbb{N} :\ m+n \in B\} = B-n$.

Noting that $(n + i)/x = \{m \in \mathbb{N} :\ m \cdot x = n + i\} = \lambda$, we have that

$$\nu((B + i)/x - (n + i)/x) = \nu(\{m \in \mathbb{N} :\ m + \lambda \in (B + i)/x\}) =$$
$$\nu(\{m \in \mathbb{N} :\ (m + \lambda) \cdot x \in B + i\}) = \nu(\{m \in \mathbb{N} :\ m \cdot x + n + i \in B + i\}) =$$
$$\nu(\{m \in \mathbb{N} :\ m \cdot x \in B - n\}) = \nu((B - n)/x) = 1.$$

Let $C = (B + i)/x = \{y \in \mathbb{N} :\ y \cdot x \in B + i\} = \{\frac{z}{x} :\ z \in (B + i) \cap \mathbb{N}x\}$, where $Ux = \{m \cdot x :\ m \in U\}$ for every $U \subseteq \mathbb{N}$. From the previous relation we have that $\nu(C - \lambda) = 1$, so, according to Lemma 9.2 there exists a finite $F \subseteq \mathbb{N}$ such that for all $y \in \mathbb{N}$ to have $\nu((\bigcup_{z\in F}(C - z)) - y) = 1$.

Pick $k \in \mathbb{N}$ such that $F \subseteq \{1, \ldots, k\}$ and let $G = \{1, 2, \ldots, (k + 1) \cdot x\}$.

Claim $(\nu \circ \mu_x)((\bigcup_{z\in G}(B - z)) - y) = 1$.

Let $y \in \mathbb{N}$ and pick $\alpha \in \mathbb{N} : (\alpha - 1) \cdot x \leq y < \alpha \cdot x$. Then, $\nu((\bigcup_{z\in F}(C - z)) - \alpha) = 1$, and since $(\bigcup_{z\in F}(C - z)) - \alpha = \bigcup_{z\in F}(C - (\alpha + z))$, we can pick $t \in F$ such that $\nu(\bigcup_{z\in F}(C - (\alpha + t)) = 1$. Then,

$$(\nu \circ \mu_x)(Cx - (\alpha + t) \cdot x) = \nu(\{m \in \mathbb{N} :\ m \cdot x \in Cx - (\alpha + t) \cdot x\}) = \nu(C - (\alpha + t)) = 1,$$

with

$$Cx - (\alpha + t) \cdot x = ((B + i)/x)x - (\alpha + t) \cdot x = \{m \in \mathbb{N} :\ m + (\alpha + t) \cdot x \in ((B + i)/x)x\} =$$
$$\{m \in \mathbb{N} :\ m + (\alpha + t) \cdot x = p \cdot x,\ p \in (B + i)/x\} \subseteq$$
$$\{m \in \mathbb{N} :\ m = p \cdot x - \alpha \cdot x - t \cdot x,\ p \cdot x \in B + i\} \subseteq B + i - \alpha \cdot x - t \cdot x,$$

so,

$$(\nu \circ \mu_x)(B + i - \alpha \cdot x - t \cdot x) = 1.$$

To finish the proof, take $s \in \{1, \ldots, x\}$ such that $\alpha \cdot x = y + s$. Note that since $t \in F \subseteq \{1, \ldots, k\}$ and $i \in \{0, 1, \ldots, x - 1\}$ we have that $1 \leq t \cdot x + s - i \leq (k + 1) \cdot x$, thus, $t \cdot x + s - i \in G$, so,

$$B + i - \alpha \cdot x - t \cdot x = B = i - y - s - t \cdot x = (B + i - s - t \cdot x) - y =$$
$$(B - (-i + s + t \cdot x)) - y \subseteq (\bigcup_{z\in G}(B - z)) - y,$$

from where we got $(\nu \circ \mu_x)((\bigcup_{z\in G}(B-z))-y)=1$. $\square$

Theorem 9.4 (Bergelson, Hindman, 1990). *For any finite partition* $\mathbb{N}=\bigcup_{i=1}^{r} C_i$, $r\in\mathbb{N}$, *there exists* $1\leq i_0\leq r$ *such that* C_{i_0} *is doth additively and multiplicatively central.*

Proof. Let $\mathcal{M}=cl\{\mu:\ \mu$ is a minimal idempotent in $(\beta\mathbb{N},*)\}$. From the previous theorem we have that $\mathcal{M}$ is a left ideal in $(\beta\mathbb{N},\circ)$. Let $I\subseteq\mathcal{M}$ be a minimal right ideal in $(\beta\mathbb{N},\circ)$ and pick an idempotent $\nu\in I$. There exists $1\leq i_0\leq r$ such that $\nu(C_{i_0})=1$, so, since ν is a minimal idempotent in $(\beta\mathbb{N},\circ)$, C_{i_0} is central in $(\mathbb{N},\cdot)$. Since $\nu\in\mathcal{M}$, with $\nu(C_{i_0})=1$, there exists a minimal idempotent μ in $(\beta\mathbb{N},*)$ satisfying $\mu(C_{i_0})=1$, hence, C_{i_0} is central in $(\mathbb{N},+)$ as well. $\square$

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