ABSTRACT TOPOLOGICAL RAMSEY THEORY FOR NETS

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Abstract. By introducing the class of coideals on an infinite directed set X, a class that contains all the adequate ultrafilters on X, it becomes possible to prove partition results for the ordered finite or infinite sequences in X with respect to a given coideal on X. Our theory extends the classical Ramsey theory, and in addition includes as particular cases (a) the corresponding theory for coideals on the set of natural numbers proved by Louveau, Mathias, Farah and Todorcevic, (b) the Milliken-Taylor partition theorems for sequences of finite subsets of natural numbers, and (c) the partition theorems for sequences of words proved by Carlson, Bergelson-Blass-Hindman, Farmaki.

1. Introduction

The starting point of the topological Ramsey theory was the Nash-Williams theorem [14], namely the statement: for a given pointwise closed partition family $\mathcal{U}$ of the topological space $[\mathbb{N}]^{\omega}_{*}$, of all the infinite strictly increasing sequences of natural numbers endowed with the relative topology of the product topology of $\{0,1\}^{\mathbb{N}}$, and every infinite subset A of $\mathbb{N}$ there exists an infinite subset B of A such that

$$\textit{either} \;\; [B]^{\omega}_{*} \subseteq \mathcal{U} \;\; \textit{or} \;\; [B]^{\omega}_{*} \subseteq [\mathbb{N}]^{\omega}_{*} \setminus \mathcal{U}.$$

It was later proved by Galvin and Prikry in [8] that the Borel partition families $\mathcal{U}$ have the same property.

These theorems were strengthened in two different ways. In one hand Louveau in [11] replaces the infinite subsets A, B by elements of a nonprincipal Ramsey ultrafilter on the set of natural numbers and, later in [10], [3] and [18], the infinite subsets A, B were replaced by elements of a semiselective coideal on the set of natural numbers.

On the other hand these topological results were extended by Milliken, Taylor, in [13], [18], proving analogous results for the set of all the finite non-empty subsets of $\mathbb{N}$ instead of the set $\mathbb{N}$ (Example 2.8, (3)) and more extended forms appeared later in [5]. Moreover, analogous topological partition results for words over a finite alphabet were proved initially by Carlson in [2] and later in [1] and [5]. Also, an extended Nash-Williams type partition theorem for the set of variable ω-located words over an infinite alphabet dominated by an increasing sequence (Example 2.8, (4)) , is proved in [4] and recently a Nash-Williams type partition theorem for the set of rational numbers is proved in [7].

In this paper we unify these two fundamentally different theories, which have as common part the classical Nash-Williams theorem, proving an abstract topological Ramsey theory for nets.

Analytically, we introduce the notions of a coideal and of a coideal basis on an infinite directed set $(X, \prec)$ (in Definitions 2.3 and 2.6 respectively). We note that the class of coideals on X contains the ultrafilters on X with cofinal with X elements and is contained in the class of the coideal bases on X. Consequently, we introduce the subclass of the

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semiselective* coideal bases on X (in Definitions 3.1 and 3.2) and we prove (in Theorem 3.6) that every semiselective* coideal basis $\mathcal{B}$ on X such that $\{y \in B : x \prec y\} \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $x \in X$ (and consequently every semiselective*coideal on X) has the Ramsey* property, whih means that for every natural number n, every partition family $\mathcal{F}$ of $[X]^n_* = \{(x_1, \ldots, x_n) : x_1 \prec \cdots \prec x_n \in X\}$ and $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that

$$\text{either } [B]^n_* \subseteq \mathcal{F} \text{ or } [B]^n_* \subseteq [X]^n_* \setminus \mathcal{F}.$$

Additionally, we prove (in Theorem 4.6), a Nash-Williams type theorem for infinite directed sets, which is a partition theorem for the set $[X]^{<\infty}_*$ of all the totally ordered finite subsets of X. The precise statement of this Theorem is as follows: for a given partition family $\mathcal{F}$ of the set $[X]^{<\infty}_*$ and every element A of a semiselective* coideal $\mathcal{B}$ on X, there exists $B \in \mathcal{B},\ B \subseteq A$ such that

either $[B]^{<\infty}_* \subseteq [X]^{<\infty}_* \setminus \mathcal{F}$,

or every infinite increasing sequence of elements of B has an initial segment in $\mathcal{F}$.

If the set $[X]^\omega_*$ of all the infinite ordered sequences in X be endowed with the relative topology of the product topology of $\{0,1\}^X$, then the Nash-Williams type theorem implies (Theorem 5.5) that for every pointwise closed partition family $\mathcal{U}$ of the topological space $[X]^\omega_*$ and every element A of a semiselective* coideal $\mathcal{B}$ on X, there exists $B \in \mathcal{B},\ B \subseteq A$ such that

$$\text{either } [B]^\omega_* \subseteq \mathcal{U} \text{ or } [B]^\omega_* \subseteq [X]^\omega_* \setminus \mathcal{U}.$$

Moreover, we prove, in Corollary 5.9, that every Borel partition family $\mathcal{U}$ of the topological space $[X]^\omega_*$ has the same property.

Since (a) the semiselective coideals on the set $\mathbb{N}$ of natural numbers are semiselective* on the directed set $\mathbb{N}$, and (b) the topological partition results, for the set of all finite non-empty subsets of $\mathbb{N}$ and also for sets with words, may be, by defining suitable semiselective* coideal bases (see Propositions 3.5 and 3.4), reformulated, it follows that these results ((a) and (b)) are included in the theory presented here, and thus unified under it.

Notation. We denote by $\mathbb{N}$ the set $\{1, 2, \ldots\}$ of all natural numbers. For a set X we denote by $[X]^{<\infty}$ the set of all finite subsets of X, by $[X]^{<\infty}_{>0}$ the set of all finite non-empty subsets of X and by $[X]^\infty$ the set of all infinite subsets of X.

2. Coideals on directed sets

In this section we first introduce, in Definitions 2.2, 2.3, the notion of a coideal on an infinite set endowed with a relation which makes it a directed set, extending the analogous notion defined earlier for the set of natural numbers in [10], [3] and [16]. Obviously, every nonprincipal ultraflter on the set of natural numbers is a coideal on it, and also every nonprincipal ultraflter $\mathcal{U}$ on an infinite directed set $(X, \prec)$ with the property that, for every $A \in \mathcal{U}$ and $x \in X$ there exists $z \in A$ with $x \prec z$, is a coideal on $(X, \prec)$. Moreover, every union of a family of such ultrafilters on X is a coideal on $(X, \prec)$.

We also introduce the notion of a coideal basis on an infinite directed set $(X, \prec)$ (in Definition 2.6). A coideal basis generates a unique coideal, which consists of all the subsets of X witch have as a subset an element of the basis. We mention some remarkable examples of coideal bases on infinite directed sets, which have a central role in Ramsey theory (in Examples 2.8). It is notable that the proof that these sets are coideal bases rests heavily on the theory of ultrafilters.

Finally, we introduce, in Definition 2.9, the notions of a Ramsey and also of a Ramsey* coideal basis on an infinite directed set. The notion of a Ramsey ultrafilter on the set of natural numbers was defined in [11], while later the analogous notion for coideals on the set of natural numbers appeared in [10], [3] and [16]. The Ramsey* property of a coideal basis on an infinite directed set is strictly weaker than the Ramsey property, but this turns out to be the suitable notion for the general case of the coideals on directed sets, since many coideals on infinite directed sets, which have central role in Ramsey theory, are Ramsey* but not Ramsey (Remarks 2.10 and Propositions 2.11 and 2.12).

Let us start with the notion of a coideal on the set of natural numbers, which appears earlier in [10], [3] and recently in [16] under the name of *superfilter*.

Definition 2.1. A subset $\mathcal{H}$ of $[\mathbb{N}]^\infty$ is a **coideal** on $\mathbb{N}$ if it satisfies the following two properties:

(i) If $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$ or $B \in \mathcal{H}$.
(ii) If $A \in \mathcal{H}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{H}$.

We will extent this notion defining the more general notion of a coideal on an infinite directed set.

Definition 2.2. Let X be a non-empty set and $\prec$ a relation on X satisfying the following conditions:

(i) If $x, y \in X$ with $x \prec y$, then $x \neq y$.
(ii) If $x, y, z \in X$ with $x \prec y$ and $y \prec z$, then $x \prec z$.
(iii) For every $x, y \in X$ there exists $z \in X$ such that $x \prec z$ and $y \prec z$.

Then $(X, \prec)$ is a **directed** set.

Definition 2.3. Let $(X, \prec)$ be an infinite directed set. A subset $\mathcal{H}$ of $[X]^\infty$ is a **coideal** on $(X, \prec)$ if it satisfies the following three properties:

(i) For every $A \in \mathcal{H}$ and $x \in X$ there exists $z \in A$ such that $x \prec z$.
(ii) If $A \cup B \in \mathcal{H}$, then either $A \in \mathcal{H}$ or $B \in \mathcal{H}$.
(iii) If $A \in \mathcal{H}$ and $A \subseteq B \subseteq X$, then $B \in \mathcal{H}$.

Throughout this paper $(X, \prec)$ will be an infinite directed set.

Notation. Let $A \subseteq X$, $s \in [X]^{<\infty}_{>0}$ and $x \in X$. Then $A - \emptyset = A$ and for $s \neq \emptyset$

$$A - s = \{x \in A : y \prec x \text{ for every } y \in s\},$$

$$A - x = A - \{x\} = \{z \in A : x \prec z\}.$$

Remarks 2.4. (i) A nonprincipal ultraflter on $\mathbb{N}$ is a coideal on $\mathbb{N}$, which is also closed under finite intersections. So, $[\mathbb{N}]^\infty$ is a coideal on $\mathbb{N}$ which is not an ultrafilter.

(ii) Let $(X, \prec)$ be an infinite directed set. An nonprincipal ultraflter $\mathcal{U}$ on X is called **adequate** if $A - x \neq \emptyset$ for every $A \in \mathcal{U}$ and $x \in X$. So, an adequate ultraflter on X is a coideal on X, which is also closed under finite intersections.

(iii) Let $(X, \prec)$ be an infinite directed set. Every union of a family of adequate ultrafilters on X is a coideal on $(X, \prec)$.

Proposition 2.5. *Let $\mathcal{H} \subseteq [X]^\infty$ be a coideal on $(X, \prec)$, $A \in \mathcal{H}$ and $s \in [X]^{<\infty}$. Then $A - s \in \mathcal{H}$.*

Proof. Let $x \in X$. We have that $A - x \in \mathcal{H}$, since $A \setminus (A - x) \notin \mathcal{H}$. For $s = \{x_1, \ldots, x_k\} \in [X]^{<\infty}$, $k \in \mathbb{N}$ we have that $A - s \in \mathcal{H}$, since $A \setminus (A - s) = A \setminus (A - x_1) \cup \ldots \cup A \setminus (A - x_k) \notin \mathcal{H}$. □

We will define now the notion of a coideal basis on an infinite directed set.

Definition 2.6. Let $(X, \prec)$ be an infinite directed set. A subset $\mathcal{B}$ of $[X]^\infty$ is a **coideal basis** on $(X, \prec)$ if it satisfies the following two properties:

(i) For every $A \in \mathcal{B}$ and $x \in X$ there exists $z \in A$ such that $x \prec z$.

(ii) If $A \cup B \in \mathcal{B}$, then there exists $C \in \mathcal{B}$ such that either $C \subseteq A$ or $C \subseteq B$.

Obviously, a coideal on $(X, \prec)$ is a coideal basis on $(X, \prec)$. The connection between coideals and coideal bases is given in the following proposition.
For $\mathcal{B} \subseteq [X]^\infty$, we set $\mathcal{L}_\mathcal{B} = \{A \subseteq X : \text{ there exists } B \in \mathcal{B} \text{ with } B \subseteq A\}$.

Proposition 2.7. *Let $\mathcal{H} \subseteq [X]^\infty$. Then $\mathcal{H}$ is a coideal on $(X, \prec)$ if and only if there exists a coideal basis $\mathcal{B} \subseteq [X]^\infty$ such that $\mathcal{H} = \mathcal{L}_\mathcal{B}$.*

Proof. If $\mathcal{B}$ is a coideal basis on $(X, \prec)$, then $\mathcal{L}_\mathcal{B}$ is obviously a coideal, since if there exists $D \in \mathcal{B}$ such that $D \subseteq A \cup B$, then $D = (D \cap A) \cup (D \cap B)$ and consequently there exists $C \in \mathcal{B}$ such that either $C \subseteq D \cap A \subseteq A$ or $C \subseteq D \cap B \subseteq B$.
If $\mathcal{H}$ be a coideal, then $\mathcal{H}$ is obviously a coideal basis and $\mathcal{H} = \mathcal{L}_\mathcal{H}$. □

We will give now some notable examples of coideal bases on directed sets

Examples 2.8. (1) The set $[\mathbb{N}]^\infty$ is a coideal on $\mathbb{N}$ with the usual order, according to the pigeon-hole principle.

(2) Let $k \in \mathbb{N}$. For $M \in [\mathbb{N}]^\infty$ let $[M]^k = \{\{m_1, \ldots, m_k\} : m_i \in M, 1 \leq i \leq k\}$. The family $\{[M]^k : M \in [\mathbb{N}]^\infty\}$ is a coideal on the set $[\mathbb{N}]^k$ endowed with the lexicographical order, according to the fundamental theorem of Ramsey ([15], 1930).

(3) Let $[\mathbb{N}]^{<\infty}_{>0}$ the set of all the finite non-empty subsets of $\mathbb{N}$. For $F_1, F_2 \in [\mathbb{N}]^{<\infty}_{>0}$ we define $F_1 \prec F_2$ if $\max F_1 < \min F_2$. Then $([\mathbb{N}]^{<\infty}_{>0}, \prec)$ is an infinite directed set.

For a sequence $(F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}$ such that $F_n \prec F_{n+1}$ for every $n \in \mathbb{N}$ we set $FU((F_n)_{n\in\mathbb{N}}) = \{\bigcup_{i\in\alpha} F_i : \alpha \in [\mathbb{N}]^{<\infty}_{>0}\}$. The family

$$\{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0} \text{ with } F_1 \prec F_2 \prec \ldots\}$$

is a coideal basis on $([\mathbb{N}]^{<\infty}_{>0}, \prec)$, according to the fundamental theorem of Hindman ([9], 1974).

(4) Let $\Sigma = \{\alpha_1, \alpha_2, \ldots\}$ be an infinite countable alphabet, $\upsilon \notin \Sigma$ a variable and $\vec{\kappa} = (\kappa_n)_{n\in\mathbb{N}} \subseteq \mathbb{N}$ an increasing sequence. The set of **variable ω-located words** over Σ dominated by the sequence $\vec{k}$ is:

$L(\Sigma, \vec{k}; \upsilon) = \{w = z_{n_1} \ldots z_{n_l} : l \in \mathbb{N},\ n_1 < \ldots < n_l \in \mathbb{N},\ z_{n_i} \in \{\upsilon, \alpha_1, \ldots, \alpha_{k_{n_i}}\}$ for all $1 \leq i \leq l$ and there exists $1 \leq i \leq l$ with $z_{n_i} = \upsilon\}$.

The set $dom(w) = \{n_1, \ldots, n_l\}$ is the *domain* of a variable ω-located word $w = z_{n_1} \ldots z_{n_l}$. For $w, u \in L(\Sigma, \vec{k}; \upsilon)$ we define $w \prec u$ if $\max dom(w) < \min dom(u)$. Then $(L(\Sigma, \vec{k}; \upsilon), \prec)$ is an infinite directed set. Let

$$L^\infty(\Sigma, \vec{k}; \upsilon) = \{\vec{w} = (w_n)_{n\in\mathbb{N}} : w_n \in L(\Sigma, \vec{k}; \upsilon) \text{ and } w_n \prec w_{n+1} \text{ for } n \in \mathbb{N}\}.$$

We will define now the extracted variable ω-located words of a sequence $\vec{w} \in L^\infty(\Sigma, \vec{k}; \upsilon)$. For $w = w_{n_1} \ldots w_{n_r}, u = u_{m_1} \ldots u_{m_l} \in L(\Sigma, \vec{k}; \upsilon)$ with $w \prec u$ we define the concatenating word $w \star u = w_{n_1} \ldots w_{n_r} u_{m_1} \ldots u_{m_l} \in L(\Sigma, \vec{k}; \upsilon)$. Also, for $w = z_{n_1} \ldots z_{n_l} \in L(\Sigma, \vec{k}; \upsilon)$ let $n_w = \min dom(w) \in \mathbb{N}$ and for every $p \in \{1, \ldots, k_{n_w}\}$ we set $w(p) = u_{n_1} \ldots u_{n_l}$, where, for $1 \leq i \leq l$, $u_{n_i} = z_{n_i}$ if $z_{n_i} \in \Sigma$, $u_{n_i} = \alpha_p$ if $z_{n_i} = \upsilon$ and we set $w(\upsilon) = w$.

Fix a sequence $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon))$. Then $u \in L(\Sigma, \vec{k}; \upsilon)$ is an **extracted variable ω-located word** of $\vec{w}$ if

$$u = w_{n_1}(p_1) \star \ldots \star w_{n_\lambda}(p_\lambda)),$$

where $\lambda \in \mathbb{N}$, $n_1 < \ldots < n_\lambda \in \mathbb{N}$, $p_i \in \{1, \ldots, k_{n_i}\} \cup \{\upsilon\}$ for every $1 \leq i \leq \lambda$ and $\upsilon \in \{p_1, \ldots, p_\lambda\}$. The set of all the extracted variable ω-located words of $\vec{w}$ is denoted by $EV(\vec{w})$. The family

$$\{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)\}$$

is a coideal basis on $(L(\Sigma, \vec{k}; \upsilon), \prec)$, according to a partition theorem for ω-located words proved in [2] and [4] and earlier in [1] for the particular case of a finite alphabet.

Notation. Let $(X, \prec)$ be an infinite directed set and $A \subseteq X$, $n \in \mathbb{N}$. We denote,

$$[A]^{<\infty}_* = \{(x_1, \ldots, x_n) : n \in \mathbb{N},\ x_1 \prec \cdots \prec x_n \in A\} \cup \{\emptyset\},$$

$$[A]^n = \{\{x_1, \ldots, x_n\} : x_1, \ldots, x_n \in A\},\ \textit{and}$$

$$[A]^n_* = \{(x_1, \ldots, x_n) : x_1 \prec \cdots \prec x_n \in A\}.$$

We introduce now the notions of the Ramsey and also of the Ramsey* coideal bases on an infinite directed set.

Definition 2.9. Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ be a coideal basis on $(X, \prec)$.

(i) $\mathcal{B}$ is **Ramsey** if for any $n, r \in \mathbb{N}$ and for $A \in \mathcal{B}$ with $[A]^n = C_1 \cup \cdots \cup C_r$ there exist $B \in \mathcal{B}$, $B \subseteq A$ and $1 \leqslant i_0 \leqslant r$ such that $[B]^n \subseteq C_{i_0}$.

(ii) $\mathcal{B}$ is **Ramsey*** if for any $n, r \in \mathbb{N}$ and for $A \in \mathcal{B}$ with $[A]^n_* = C_1 \cup \cdots \cup C_r$ then there exist $B \in \mathcal{B}$, $B \subseteq A$ and $1 \leqslant i_o \leqslant r$ such that $[B]^n_* \subseteq C_{i_0}$.

Remarks 2.10. Let $(X, \prec)$ be an infinite directed set.

(i) If a coideal basis on X is Ramsey, then it is obviously Ramsey*.

(ii) If $(X, \prec)$ is totally ordered set, then a coideal basis on X is Ramsey if and only if it is Ramsey*, since $[A]^n = [A]^n_*$ for every $n \in \mathbb{N}$ and $A \in \mathcal{B}$.

(iii) The coideal basis $\{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)\}$, as proved in [4] and in [1] for the case of a finite alphabet, is Ramsey* on $(L(\Sigma, \vec{k}; \upsilon), \prec)$, but it is not Ramsey according to the following proposition.

Proposition 2.11. *The coideal basis* $\mathcal{B} = \{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)\}$ *on* $(L(\Sigma, \vec{k}; \upsilon), \prec)$ *is not Ramsey.*

Proof. Let $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ and let $[EV(\vec{w})]^2 = C_1 \cup C_2$, where

$$C_1 = \{(w_1, w_2) \in [EV(\vec{w})]^2 : w_1 \prec w_2\}, \text{ and } C_2 = [EV(\vec{w})]^2 \setminus C_1.$$

We suppose that $\mathcal{B}$ is Ramsey and let a sequence $\vec{u} = (u_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ such that either $[EV(\vec{u})]^2 \subseteq C_1$ or $[EV(\vec{u})]^2 \subseteq C_2$. Since $(u_1, u_1 * u_2) \notin C_1$ we have that the first alternative does not hold. Also, since $(u_1, u_2) \notin C_2$ and the second alternative does not hold, so we have a contradiction. Hence, $\mathcal{B}$ is not Ramsey. □

Proposition 2.12. *The coideal basis* $\{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}, F_1 \prec F_2 \prec \ldots\}$ *on* $([\mathbb{N}]^{<\infty}_{>0}, \prec)$ *is Ramsey*, but it is not Ramsey.*

Proof. It is proved by Milliken, Taylor in [13], [17] that the coideal basis $\{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}$ with $F_1 \prec F_2 \prec \ldots\}$ is Ramsey*. That the coideal basis is not Ramsey can be proved analogously to the previous proposition. □

3. Semiselective* coideal bases and Ramsey property

The notion of a semiselective coideal on the set of natural numbers was introduced by Farah in [3], where some interesting characterizations for this notion are proved(see also [18]). We will introduce now, in Definitions 3.1 and 3.2, the notion of a semiselective* coideal basis on an infinite directed set, extending the previous notion defined for the set of natural numbers.

We prove, in Theorem 3.6, that every semiselective* coideal basis on an infinite directed set $(X, \prec)$, with the property that $B - x \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $x \in X$, is Ramsey*; consequently every semiselective* coideal on an infinite directed set is Ramsey*.

In Propositions 3.4 and 3.5 below, we prove that the coideal bases mentioned in Examples 2.8, (3) and (4) on the set of the finite sets of natural numbers and on the set of variable ω-located words, respectively, are semiselective* and in addition satisfy the second property. So we have another proof that these bases are Ramsey*.

Definition 3.1. Let $(X, \prec)$ be an infinite directed set, $\mathcal{B} \subseteq [X]^\infty$ be a coideal basis on X and $A \in \mathcal{B}$. A set $\mathcal{R} \subseteq \mathcal{B}$ has the **dense-open property** in $\mathcal{B}$ (resp. has the **dense-open property** in $\mathcal{B}$ on A) if the following two conditions hold:

(i) For every $B \in \mathcal{B}$, (resp. for every $B \in \mathcal{B}$, $B \subseteq A$) there exists $C \in \mathcal{R}$ with $C \subseteq B$.
(ii) For every $C \in \mathcal{R}$ and for every $B \in \mathcal{B}$ with $B \subseteq C$ we have that $B \in \mathcal{R}$.

Definition 3.2. Let $(X, \prec)$ be an infinite directed set. A coideal basis $\mathcal{B} \subseteq [X]^\infty$ on $(X, \prec)$ is **semiselective*** if for every family $(\mathcal{R}_s)_{s \in [X]^{<\infty}_*}$, where $\mathcal{R}_s \subseteq \mathcal{B}$ has the dense-open property in $\mathcal{B}$, and for every $B \in \mathcal{B}$ there exists $B' \in \mathcal{B}$, $B' \subseteq B$ such that for every $s \in [B']^{<\infty}_*$ and $\widetilde{B} \subseteq B' - s$, $\widetilde{B} \in \mathcal{B}$ we have that $\widetilde{B} \in \mathcal{R}_s$.

Proposition 3.3. *Let $(X, \prec)$ be an infinite directed set, $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on $(X, \prec)$ and $A \in \mathcal{B}$. Then for every family $(\mathcal{R}^A_s)_{s \in [X]^{<\infty}_*}$, where $\mathcal{R}^A_s \subseteq \mathcal{B}$ has the dense-open property in $\mathcal{B}$ on A, there exists $B' \in \mathcal{B}$, $B' \subseteq A$ such that for every $s \in [B']^{<\infty}_*$ and $\widetilde{B} \subseteq B' - s$, $\widetilde{B} \in \mathcal{B}$ we have that $\widetilde{B} \in \mathcal{R}^A_s$.*

Proof. Let the family $(\mathcal{R}^A_s)_{s \in [X]^{<\infty}_*}$, where $\mathcal{R}^A_s \subseteq \mathcal{B}$ has the dense-open property in $\mathcal{B}$ on A for every $s \in [X]^{<\infty}_*$. We define for $s \in [X]^{<\infty}_*$ the set

$$\mathcal{R}_s = \mathcal{R}^A_s \cup \{B \in \mathcal{B} : \nexists\, C \in \mathcal{B},\ C \subseteq A \cap B\}.$$

The set $\mathcal{R}_s$ has the dense-open property in $\mathcal{B}$. Indeed,
(i) Let $B \in \mathcal{B}$. If there exists $C \in \mathcal{B}$, $C \subseteq A \cap B$, then there exists $D \in \mathcal{R}^A_s \subseteq \mathcal{R}_s$ with $D \subseteq C \subseteq B$. Otherwise, $B \in \mathcal{R}_s$.
(ii) Let $C \in \mathcal{R}_s$ and $B \in \mathcal{B}$ with $B \subseteq C$. If $C \in \mathcal{R}^A_s$, then $B \in \mathcal{R}^A_s \subseteq \mathcal{R}_s$. If $C \notin \mathcal{R}^A_s$, then does not exist $D \in \mathcal{B}$, $D \subseteq A \cap C$. Hence, $B \in \mathcal{R}_s$.

Since $\mathcal{B}$ is semiselective* coideal basis, there exists $B' \in \mathcal{B}$, $B' \subseteq A$ such that for every $s \in [B']^{<\infty}_*$ and $\widetilde{B} \subseteq B' - s$, $\widetilde{B} \in \mathcal{B}$ we have that $\widetilde{B} \in \mathcal{R}^A_s$. □

Notation. For $w_1, \dots, w_k \in L(\Sigma, \vec{k}; \upsilon)$ with $w_1 \prec \dots \prec w_k$ we write
$EV(w_1, \dots, w_k) = \{w_{n_1}(p_1) \star \dots \star w_{n_\lambda}(p_\lambda) : \lambda \in \mathbb{N}, n_1 < \dots < n_\lambda \leq k \in \mathbb{N},\ p_i \in \{1, \dots, k_{n_i}\} \cup \{\upsilon\}$ for every $1 \leq i \leq \lambda$ and $\upsilon \in \{p_1, \dots, p_\lambda\}\}$.
Observe that the set $EV(w_1, \dots, w_k)$ is finite.

Proposition 3.4. *The coideal basis*
$$\mathcal{B} = \{EV(\vec{w}) : \vec{w} = (w_n)_{n \in \mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)\} \text{ on } (L(\Sigma, \vec{k}; \upsilon), \prec)$$

is semiselective and* $B - s \in \mathcal{B}$ *for every* $B \in \mathcal{B}$ *and* $s \in [L(\Sigma, \vec{k}; \upsilon)]_*^{<\infty}$.

Proof. Let $EV(\vec{w}) \in \mathcal{B}$, where $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ and $s \in [L(\Sigma, \vec{k}; \upsilon)]_*^{<\infty}$, $s \neq \emptyset$. Then $EV(\vec{w}) - s = EV(\vec{w} - s) \in \mathcal{B}$, where $\vec{w} - s = (w_n)_{n=n_0}^\infty \in L^\infty(\Sigma, \vec{k}; \upsilon)$ with $n_0 = \min\{n \in \mathbb{N} : \min dom(w_n) > \max s\}$.

Let $(\mathcal{R}_s)_{s\in[L(\Sigma,\vec{k};\upsilon)]_*^{<\infty}}$, where $\mathcal{R}_s \subseteq \mathcal{B}$ has the dense-open property in $\mathcal{B}$ for every $s \in [L(\Sigma, \vec{k}; \upsilon)]_*^{<\infty}$, and let $EV(\vec{w}) \in \mathcal{B}$, where $\vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$.

According to condition (i) of Definition 3.2, there exists $\vec{s}_1 = (s_n^1)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)$ with $EV(\vec{s}_1) \subseteq EV(\vec{w})$ such that $EV(\vec{s}_1) \in \mathcal{R}_\emptyset$.

We assume now that there have been constructed $\vec{s}_1, \ldots, \vec{s}_k \in L^\infty(\Sigma, \vec{k}; \upsilon)$ with $EV(\vec{s}_k) \subseteq \cdots \subseteq EV(\vec{s}_1) \subseteq EV(\vec{w})$ and let $\vec{s}_i = (s_n^i)_{n\in\mathbb{N}}$ for all $i \in \{1, \ldots, k\}$. We will construct $\vec{s}_{k+1}$. Let $\{t_1, \ldots, t_\lambda\} = EV(s_1^1, \ldots, s_k^k)$ According to condition (i) of Definition 3.2, there exist $\vec{s}_{k+1}^1, \ldots, \vec{s}_{k+1}^\lambda \in L^\infty(\Sigma, \vec{k}; \upsilon)$ such that $EV(\vec{s}_{k+1}^\lambda) \subseteq \cdots \subseteq EV(\vec{s}_{k+1}^1) \subseteq EV(\vec{s}_k - s_k^k)$ and $EV(\vec{s}_{k+1}^i) \in \mathcal{R}_{t_i}$ for every $1 \leq i \leq \lambda$. Set $\vec{s}_{k+1} = \vec{s}_{k+1}^\lambda$ and let $\vec{s}_{k+1} = (s_n^{k+1})_{n\in\mathbb{N}}$. According to condition (ii) of Definition 3.2, $EV(\vec{s}_{k+1}) \in \mathcal{R}_{t_i}$ for all $1 \leq i \leq \lambda$.

Set $\vec{s} = (s_1^1, s_2^2, \ldots) \in L^\infty(\Sigma, \vec{k}; \upsilon)$. Then $EV(\vec{s}) \subseteq EV(\vec{w})$, since $s_1^1 \prec s_2^2 \prec \ldots \in EV(\vec{w})$. Let $s \in [EV(\vec{s})]_*^{<\infty}$ and $EV(\vec{u}) \subseteq EV(\vec{s}) - s = EV(\vec{s} - s)$. Let $s \neq \emptyset$. Then we set $n_0 = \min\{n \in \mathbb{N} : s \in EV((s_1^1, \ldots, s_n^n))\}$. Since $s \in [EV((s_1^1, \ldots, s_{n_0}^{n_0}))]_*^{<\infty}$, we have $EV(\vec{s}_{n_0+1}) \in \mathcal{R}_s$. Then, according to (ii) of Definition 3.2, we have $EV(\vec{s} - s_{n_0}^{n_0}) \in \mathcal{R}_s$, since $EV(\vec{s} - s_{n_0}^{n_0}) \subseteq EV(\vec{s}_{n_0+1})$, and also $EV(\vec{u}) \in \mathcal{R}_s$, since $EV(\vec{u}) \subseteq EV(\vec{s} - s_{n_0}^{n_0}) = EV(\vec{s} - s)$. □

Proposition 3.5. *The coideal basis*

$$\mathcal{B} = \{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]_{>0}^{<\infty} \text{ with } F_n \prec F_{n+1} \text{ for every } n \in \mathbb{N}\},$$

on $([\mathbb{N}]_{>0}^{<\infty}, \prec)$, *is semiselective* and* $B - s \in \mathcal{B}$ *for every* $B \in \mathcal{B}$ *and* $s \in [[\mathbb{N}]_{>0}^{<\infty}]^{<\infty}$.

Proof. The proof is analogous to this of the previous proposition. □

We will prove now that every semiselective* coideal basis on an infinite directed set X such that $B - x \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $x \in X$ and consequently every semiselective* coideal on an infinite directed set is Ramsey*.

Notation. Let $(X, \prec)$ be an infinite directed infinite set. For a family $\mathcal{F} \subseteq [X]_*^{<\infty}$ and $y \in X$ we define

$$\mathcal{F} - y = \{s \in \mathcal{F} : \textit{ either } s = \emptyset \textit{ or } s = (x_1, \ldots, x_n) \textit{ for } n \in \mathbb{N} \textit{ and } y \prec x_1\}, \textit{ and}$$

$$\mathcal{F}(y) = \{s \in [X]_*^{<\infty} : \textit{either } s = \emptyset,\ \{y\} \in \mathcal{F} \textit{ or } s = (x_1, \ldots, x_n) \neq \emptyset,\ (y, x_1, \ldots, x_n) \in \mathcal{F}\}.$$

Theorem 3.6. *Let* $(X, \prec)$ *be an infinite directed set,* $\mathcal{B} \subseteq [X]^\infty$ *a semiselective* coideal basis on* $(X, \prec)$ *such that* $B - x \in \mathcal{B}$ *for every* $B \in \mathcal{B}$ *and* $x \in X$ *and* $\kappa \in \mathbb{N}$. *For every* $A \in \mathcal{B}$ *and* $\mathcal{F} \subseteq [X]_*^{<\infty}$ *there exists* $B \in \mathcal{B}$, $B \subseteq A$ *such that*

$$\textit{either } [B]_*^\kappa \subseteq \mathcal{F} \textit{ or } [B]_*^\kappa \subseteq [X]_*^{<\infty} \setminus \mathcal{F}.$$

Proof. We proceed by induction on κ. For $\kappa = 1$ the theorem is valid since $\mathcal{B}$ is a coideal basis and for every $A \in \mathcal{B}$ and $\mathcal{F} \subseteq [X]_*^{<\infty}$

$$A = (\mathcal{F} \cap A) \cup \left(([X]_*^{<\infty} \setminus \mathcal{F}) \cap A\right).$$

Assume that the theorem holds for some $\kappa \in \mathbb{N}$. Let $A \in \mathcal{B}$ and $\mathcal{F} \subseteq [X]_*^{<\infty}$. For every $x \in X$ we set,

$$\mathcal{R}_{\{x\}} = \{\widetilde{B} \in \mathcal{B} : \widetilde{B} \subseteq X - x \text{ and either } [\widetilde{B}]_*^\kappa \subseteq \mathcal{F}(x) \text{ or } [\widetilde{B}]_*^\kappa \subseteq [X]_*^{<\infty} \setminus \mathcal{F}(x)\}$$

and

$$\mathcal{R}_s = \mathcal{B} \text{ for every } s \in [X]^{<\infty}_* \setminus \{\{x\} : x \in X\}.$$

For $x \in X$, and $B \in \mathcal{B}$, by the induction hypothesis, there exists $\widetilde{B} \in \mathcal{B}$, $\widetilde{B} \subseteq B - x$ such that

$$\text{either } [\widetilde{B}]^{\kappa}_* \subseteq \mathcal{F}(x) \text{ or } [\widetilde{B}]^{\kappa}_* \subseteq [X]^{<\infty}_* \setminus \mathcal{F}(x).$$

So, $\widetilde{B} \in \mathcal{R}_{\{x\}}$. Also, for every $B \in \mathcal{B}$ and for every $C \in \mathcal{R}_{\{x\}}$ with $B \subseteq C$ we have that $B \in \mathcal{R}_{\{x\}}$.
We note that, $B - x \in \mathcal{B}$ for every $x \in X$ and $B \in \mathcal{B}$. By the above arguments the families $\mathcal{R}_s$, for $s \in [X]^{<\infty}_*$ have the dense-open property in $\mathcal{B}$ and, since $\mathcal{B}$ is a semiselective* coideal basis, there exists $B' \in \mathcal{B}$, $B' \subseteq A$ such that for every $x \in B'$ and $\widetilde{B} \subseteq B' - x$, $\widetilde{B} \in \mathcal{B}$ we have that $\widetilde{B} \in \mathcal{R}_x$. So, $B' - x \in \mathcal{R}_x$ for every $x \in B'$. Let

$$B_1 = \{x \in B' : [B' - x]^{\kappa}_* \subseteq \mathcal{F}(x)\}, \textit{ and}$$

$$B_2 = \{x \in B' : [B' - x]^{\kappa}_* \subseteq [X]^{<\infty}_* \setminus \mathcal{F}(x)\}.$$

Since $B' \in \mathcal{B}$, $B' = B_1 \cup B_2$ and $\mathcal{B}$ is a coideal basis, we have that there exists $B \in \mathcal{B}$ such that either $B \subseteq B_1$ or $B \subseteq B_2$ and of course $B \subseteq A$. In case $B \subseteq B_1$, we have that $[B - x]^{\kappa}_* \subseteq \mathcal{F}(x)$ for every $x \in B$ and consequently that $[B]^{\kappa+1}_* \subseteq \mathcal{F}$. Also, in case $B \subseteq B_2$, we have that $[B - x]^{\kappa}_* \subseteq [X]^{<\infty}_* \setminus \mathcal{F}(x)$ for every $x \in B$ and consequently that $[B]^{\kappa+1}_* \subseteq [X]^{<\infty}_* \setminus \mathcal{F}$. The proof is complete. □

The above partition theorem implies the existence of more coideal bases.

Corollary 3.7. *Let $(X, \prec)$ be an infinite directed set, $\mathcal{B} \subseteq [X]^{\infty}$ a semiselective* coideal basis on $(X, \prec)$ such that $B - x \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $x \in X$. The family $\{[A]^n_* : A \in \mathcal{B}\}$ is a coideal basis on $([X]^n_*, \prec)$, where, for $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in [X]^n_*$, we define $(x_1, \ldots, x_n) \prec (y_1, \ldots, y_n)$ if $x_n \prec y_1$.*

Theorem 3.6 implies that every semiselective* coideal $\mathcal{H}$ on an infinite directed set $(X, \prec)$ is Ramsey*.

Corollary 3.8. *Let $(X, \prec)$ be an infinite directed set, $\mathcal{H} \subseteq [X]^{\infty}$ a semiselective* coideal on $(X, \prec)$ and let $\kappa \in \mathbb{N}$. For every $A \in \mathcal{B}$ and $\mathcal{F} \subseteq [X]^{<\infty}_*$ there exists $B \in \mathcal{H}$, $B \subseteq A$ such that*

$$\textit{either } [B]^{\kappa}_* \subseteq \mathcal{F} \textit{ or } [B]^{\kappa}_* \subseteq [X]^{<\infty}_* \setminus \mathcal{F}.$$

The coideal basis $\{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^{\infty}(\Sigma, \vec{k}; \upsilon)\}$ on $(L(\Sigma, \vec{k}; \upsilon), \prec)$ is semiselective*, according to Proposition 3.4, and $B - x \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $x \in L(\Sigma, \vec{k}; \upsilon)$. Hence, from Theorem 3.6 follows that it is Ramsey*, a result that is proved in [4].

Corollary 3.9 ([4])**.** *The coideal basis $\mathcal{B} = \{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^{\infty}(\Sigma, \vec{k}; \upsilon)\}$ on $(L(\Sigma, \vec{k}; \upsilon), \prec)$ is Ramsey*.*

The following corollary of Theorem 3.6 is proved in [13] and [18].

Corollary 3.10 ([13], [18])**.** *The coideal basis $\mathcal{B} = \{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}$ with $F_n \prec F_{n+1}$ for every $n \in \mathbb{N}\}$ on $([\mathbb{N}]^{<\infty}_{>0}, \prec)$ is Ramsey*.*

Proof. The coideal basis $\mathcal{B}$ is semiselective*, according to Proposition 3.5, and $B - x \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $x \in [\mathbb{N}]^{<\infty}_{>0}$. So, according to Theorem 3.6, $\mathcal{B}$ is Ramsey*. □

4. A Nash-Williams type theorem for coideals on directed sets

In this section we will prove a partition theorem (in Theorem 4.6) for the set of all the totally ordered finite subsets of an infinite directed set. Precisely, we will prove that for a given partition family $\mathcal{F}$ of the set of all the non-empty, totally ordered, finite subsets of an infinite directed set X and for a given element A of a semiselective* coideal basis $\mathcal{B}$ on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]^{<\infty}_*$ there exists $B \in \mathcal{B}$, $B \subseteq A$ such that,

either all the totally ordered finite subsets of B to be elements of the complement of $\mathcal{F}$,

or every infinite increasing sequence of elements of B to have an initial segment in $\mathcal{F}$.

This partition theorem is called a Nash-Williams type theorem for $\mathcal{B}$ on X, as the particular case of this theorem for $\mathcal{B} = [\mathbb{N}]^\infty$ on $X = \mathbb{N}$ firstly has been proved by Nash-Williams in [14].

This partition theorem applied to the set $[\mathbb{N}]^{<\infty}_{>0}$ of all the finite non-empty subsets of $\mathbb{N}$ for the coideal basis $\mathcal{B} = \{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]^{<\infty}_{>0}$ with $F_1 \prec F_2 \prec \ldots\}$ (Example 2.8, (3), Proposition 3.5) gives the Nash-Williams type theorem on $[\mathbb{N}]^{<\infty}_{>0}$ firstly proved in [13], [17].

Also, this partition theorem implies the Nash-Williams type theorem on the set $L(\Sigma, \vec{k}; \upsilon)$ of the variable ω-located words over an infinite alphabet Σ dominated by an increasing sequence $\vec{k}$ for the coideal basis $\{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)\}$ with elements sets consisted of all the extracted variable ω-located words of an infinite ordered sequence in $(L(\Sigma, \vec{k}; \upsilon)$ (Example 2.8, (4), Proposition 3.4) proved in [4] and in case of a finite alphabet in [2] and [1].

A reformulation of Theorem 4.6 is given in Theorem 4.7, which has central role in the following section, where we will introduce a topological Ramsey theory for directed sets.

Let us start with the necessary notation.

Notation. Let $(X, \prec)$ be an infinite directed set.

(1) For $t, s \in [X]^{<\infty}_*$, we write $t \prec s$ if either $\emptyset \in \{t, s\}$, or $t, s \neq \emptyset$ and $x \prec y$ for every $x \in t$ and $y \in s$.

(2) For $t, s \in [X]^{<\infty}_*$ we write $t \sqsubseteq s$ if t is an initial segment of s, or more precisely if either $t \in \{\emptyset, s\}$ or there exists $y \in s$ such that $t = \{x \in s : x \prec y\}$.

(3) For $A \subseteq X$ we set
$$[A]_* = \{(x_n)_{n\in\mathbb{N}} \subseteq A : x_n \prec x_{n+1} \text{ for every } n \in \mathbb{N}\}.$$

(4) For $(x_n)_{n\in\mathbb{N}} \in [X]_*$ and $s \in [X]^{<\infty}_*$ we write $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$ if either $s = \emptyset$ or $s = (x_1, \ldots, x_m)$ for some $m \in \mathbb{N}$.

(5) Let $t, s \in [X]^{<\infty}_*$ with $t \prec s$. If $t = (x_1, \ldots, x_n)$, $n \in \mathbb{N}$ and $s = (s_1, \ldots, s_m)$, $m \in \mathbb{N}$, we set $t \otimes s = (x_1, \ldots, x_n, y_1, \ldots, y_m)$. If $t = \emptyset$, we set $t \otimes s = s$ and if $s = \emptyset$, we set $t \otimes s = t$.

Definition 4.1. Let $(X, \prec)$ be an infinite directed set. For every $A \in [X]^\infty$ and $t \in [X]^{<\infty}_*$ we set,

$[t, A]_* = \{(x_n)_{n\in\mathbb{N}} \in [X]_* : t \sqsubseteq (x_n)_{n\in\mathbb{N}} \text{ and } (x_n)_{n\in\mathbb{N}} - t \in [A]_*\}$,

$[\emptyset, A]_* = [A]_*$, and

$[t, A]^{<\infty}_* = \{t_1 \in [X]^{<\infty}_* : t_1 \sqsubseteq t\} \cup \{t \otimes s : s \in [A]^{<\infty}_* \textit{ and } t \prec s\}$.

Definition 4.2. Let $(X, \prec)$ be an infinite directed set, $\mathcal{B} \subseteq [X]^\infty$ be a coideal basis on X, $A \in \mathcal{B}$, $t \in [X]^{<\infty}_*$ and $\mathcal{F} \subseteq [X]^{<\infty}_* \setminus \{\emptyset\}$.

(i) We say that A **accepts** t if for every $(x_n)_{n\in\mathbb{N}} \in [t, A]_*$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

(ii) We say that A **rejects** t if there is no $B \in \mathcal{B},\ B \subseteq A$ accepting t.
(iii) We say that A **decides** t if either A accepts t or rejects t.

Remark 4.3. *Let $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on $(X, \prec)$ such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$ and $\mathcal{F} \subseteq [X]_*^{<\infty} \setminus \{\emptyset\}$. Then $A \in \mathcal{B}$ accepts (rejects) $t \in [X]_*^{<\infty}$ if and only if $A - t$ accepts (rejects) t.*

Lemma 4.4. *Let $\mathcal{B} \subseteq [X]^\infty$ be a coideal basis on X and $\mathcal{F} \subseteq [X]_*^{<\infty} \setminus \{\emptyset\}$.*
(i) *If $A \in \mathcal{B}$ accepts (rejects) $t \in [X]_*^{<\infty}$, then every $B \in \mathcal{B},\ B \subseteq A$ accepts (rejects) t.*
(ii) *For every $A \in \mathcal{B}$ and $t \in [X]_*^{<\infty}$ there exists $B \in \mathcal{B},\ B \subseteq A$ which decides t.*
(iii) *If $A \in \mathcal{B}$ accepts $t \in [X]_*^{<\infty}$ then A accepts $t_x = t \cup \{x\}$, for every $x \in A - t$.*
(iv) *If $A \in \mathcal{B}$ rejects $t \in [X]_*^{<\infty}$, then*

$$C = \{x \in A - t : A \ \textit{accepts} \ t \cup \{x\}\} \notin \mathcal{L}_{\mathcal{B}}.$$

Proof. (i) It follows obviously from the definitions.
(ii) Let $A \in \mathcal{B}$ and $t \in [X]_*^{<\infty}$. If A rejects t, then A decides t. Otherwise, there exists $B \in \mathcal{B},\ B \subseteq A$ which accepts t and therefore B decides t.
(iii) Let $x \in A - t$ and $(x_n)_{n\in\mathbb{N}} \in [t_x, A]_*$. Then $(x_n)_{n\in\mathbb{N}} \in [t, A]_*$. Since A accepts t, there exists $s \in \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$. So, A accepts t_x.
(iv) We assume that $C \in \mathcal{L}_{\mathcal{B}}$. Then there exists $\widetilde{C} \in \mathcal{B}$ such that $\widetilde{C} \subseteq C$. Since $\widetilde{C} \subseteq A$, according to (i), $\widetilde{C}$ rejects t.

Let $(x_n)_{n\in\mathbb{N}} \in [t, \widetilde{C}]_*$. Then there exists $x_{n_0} \in \widetilde{C} - t$, $n_0 \in \mathbb{N}$, with the property $t_{x_{n_0}} = t \cup \{x_{n_0}\} \sqsubseteq (x_n)_{n\in\mathbb{N}}$. Since $x_{n_0} \in \widetilde{C} \subseteq C$, we have that A accepts $t_{x_{n_0}}$ and consequently that there exists $s \in \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$. Hence $\widetilde{C}$ accepts t. A contradiction, since A rejects t and $\widetilde{C} \subseteq A$. □

Lemma 4.5. *Let $\mathcal{B} \subseteq [X]^\infty$ be a semiselective* coideal basis on $(X, \prec)$ such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$ and $\mathcal{F} \subseteq [X]_*^{<\infty} \setminus \{\emptyset\}$. For every $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that B decides every $t \in [B]_*^{<\infty}$.*

Proof. For every $s \in [X]_*^{<\infty}$ we set

$$\mathcal{R}_s = \{B' \in \mathcal{B} : B' \ \text{decides} \ s\}.$$

By (i) and (ii) of Lemma 4.4, the family $\mathcal{R}_s$, for $s \in [X]_*^{<\infty}$, has the dense-open property. Since $\mathcal{B}$ is semilective* coideal basis, for every $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that for every $s \in [B]_*^{<\infty}$ and $\widetilde{B} \subseteq B - s$, $\widetilde{B} \in \mathcal{B}$ we have $\widetilde{B} \in \mathcal{R}_s$. So, for every $s \in [B]_*^{<\infty}$ we have that $B - s \in \mathcal{B}$ and that $B - s$ decides s. Hence, B decides every $s \in [B]_*^{<\infty}$. □

Theorem 4.6. *Let $(X, \prec)$ be an infinite directed set, $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$ and $\mathcal{F} \subseteq [X]_*^{<\infty} \setminus \{\emptyset\}$. For every $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that*
either $[B]_*^{<\infty} \cap \mathcal{F} = \emptyset$,
or for every $(x_n)_{n\in\mathbb{N}} \in [B]_*$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

Proof. Let $A \in \mathcal{B}$. According to Lemma 4.5, there exists $B \in \mathcal{B},\ B \subseteq A$ which decides every $s \in [B]_*^{<\infty}$. If B accepts $\emptyset$, then the the second condition of this theorem holds.

Let B rejects $\emptyset$. We define the families $(\mathcal{L}_s)_{s\in[X]_*^{<\infty}}$ as follows: For $s \in [B]_*^{<\infty}$ such that B rejects s we set

$$\mathcal{L}_s = \{C \in \mathcal{B} : C \ \text{rejects} \ s_x = s \cup \{x\}, \ \text{for every} \ x \in C - s\}$$

and $\mathcal{L}_s = \mathcal{B}$ otherwise.

We will prove now that the families $\mathcal{L}_s$, for $s \in [X]^{<\infty}_*$ have the dense-open property in B. Let $s \in [B]^{<\infty}_*$ such that B rejects s.

(i) Let $B_1 \in \mathcal{B},\ B_1 \subseteq B$. Then, according to (i) of Lemma 4.4, B_1 rejects s and according to (v) of Lemma 4.4 we have that

$$C_1 = \{x \in B_1 - s : B_1 \text{ accepts } s_x = s \cup \{x\}\} \notin \mathcal{L}_{\mathcal{B}}.$$

Since B decides every $s \in [B]^{<\infty}_*$, according to Remark 4.3 and (i) of Lemma 4.4, for every $s \in [B]^{<\infty}_*$ we have that $B - s$ and consequently $B_1 - s$ decides s. Hence, $B_1 - s = C_1 \cup C_2$, where

$$C_2 = \{x \in B_1 - s : B_1 \text{ rejects } s_x = s \cup \{x\}\}.$$

Since $B_1 - s \in \mathcal{B}$ and $C_1 \notin \mathcal{L}_{\mathcal{B}}$, we have that $C_2 \in \mathcal{L}_{\mathcal{B}}$. Hence, there exists $C \in \mathcal{B}$, $C \subseteq C_2 \subseteq B_1$ and $C \in \mathcal{L}_s$.

(ii) Let $B_1 \in \mathcal{B}$ and $C \in \mathcal{L}_s$ with $B_1 \subseteq C$. Then $C \in \mathcal{B}$ and C rejects every $s_x = s \cup \{x\}$ for $x \in C - s$. Hence, $B_1 \in \mathcal{L}_s$, since $B_1 \in \mathcal{B}$ and $B_1 \subseteq C$.

Since $\mathcal{B}$ is semiselective* there exists $B' \in \mathcal{B},\ B' \subseteq B$ such that for every $s \in [B']^{<\infty}_*$ and $\widetilde{B} \subseteq B' - s,\ \widetilde{B} \in \mathcal{B}$ we have that $\widetilde{B} \in \mathcal{L}_s$. We will prove that B' rejects every $s \in [B']^{<\infty}_*$.

Indeed, we will use induction on the size $|s|$ of s. If $|s| = 0$, then $s = \emptyset$ and B' rejects $\emptyset$, since B rejects $\emptyset$ and $B' \subseteq B$. We assume that B' rejects every $s \in [B']^n_*$. Let $s \cup \{x\} \in [B']^{n+1}_*$, where $s \in [B']^n_*$ and $x \in B' - s$. Since $B' - s \in \mathcal{L}_s$ and B' rejects s by the induction hypothesis, we have that $B' - s$ rejects $s \cup \{x\}$ and consequently that B' rejects $s \cup \{x\}$. Hence, B' rejects every $s \in [B']^{<\infty}_*$.

If $s \in [B']^{<\infty}_* \cap \mathcal{F}$, then B' accepts s, thus $[B']^{<\infty}_* \cap \mathcal{F} = \emptyset$. This finishes the proof of the theorem. □

We will prove now an equivalent form of Theorem 4.6.

Theorem 4.7. *Let $(X, \prec)$ be an infinite directed set, $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]^{<\infty}_*$, let $t \in [X]^{<\infty}_*$ and $\mathcal{F} \subseteq [X]^{<\infty}_* \setminus \{\emptyset\}$. For every $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that*
either $[t, B]^{<\infty}_* \cap \mathcal{F} = \emptyset$,
or for every $(x_n)_{n\in\mathbb{N}} \in [t, B]_*$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

Proof. If $t = \emptyset$, then the theorem is identified with Theorem 4.6. We assume that $t \neq \emptyset$. Let $A \in \mathcal{B}$. If there exists $t_1 \sqsubseteq t$ with $t_1 \in \mathcal{F}$ we have obviously the condition (2) of the theorem. Let $t_1 \notin \mathcal{F}$ for every $t_1 \sqsubseteq t$. We set

$$\mathcal{F}_1 = \{s \in \mathcal{F} : t \sqsubseteq s\} \qquad \textit{and} \qquad \mathcal{F}_2 = \{(s - t) : s \in \mathcal{F}_1\}.$$

Observe that if $s \in \mathcal{F}_2$, then $t \otimes s \in \mathcal{F}_1$ and $\emptyset \notin \mathcal{F}_2$. According to Theorem 4.6, there exists $B \in \mathcal{B},\ B \subseteq A - t$ such that
either $[B]^{<\infty}_* \cap \mathcal{F}_2 = \emptyset$,
or for every $(x_n)_{n\in\mathbb{N}} \in [B]_*$ there exists $s \in \mathcal{F}_2$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.
Then,
either $[t, B]^{<\infty}_* \cap \mathcal{F} = \emptyset$,
or for all $(x_n)_{n\in\mathbb{N}} \in [t, B]_*$ there exists $s \in \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$. □

Theorem 4.7 implies the following theorem for a semiselective* coideal on an infinite directed set.

Theorem 4.8. *Let $(X, \prec)$ be an infinite directed set, $\mathcal{H} \subseteq [X]^\infty$ a semiselective* coideal on $(X, \prec)$, $t \in [X]^{<\infty}_*$ and $\mathcal{F} \subseteq [X]^{<\infty}_* \setminus \{\emptyset\}$. For every $A \in \mathcal{H}$ there exists $B \in \mathcal{H},\ B \subseteq A$ such that*

either $[t,B]_*^{<\infty} \cap \mathcal{F} = \emptyset$,
or for every $(x_n)_{n\in\mathbb{N}} \in [t,B]_*$ *there exists* $s \in \mathcal{F}$ *such that* $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

5. Topological Ramsey theory for directed sets

In this section we start with an infinite directed set $(X, \prec)$ and endow the set $[X]_*^\omega$ of all the infinite ordered sequences in X with the relative topology of the product topology of $\{0,1\}^X$. Our main aim is to locate topological conditions for a partition family $\mathcal{U}$ of $[X]_*$ in order to be $\mathcal{B}$-completely Ramsey*, which means that for a given element A of a semiselective* coideal basis $\mathcal{B}$ on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$ to exist an element $B \in \mathcal{B}$ with $B \subseteq A$ such that

either all the infinite ordered sequences in B to be elements of $\mathcal{U}$,
or all the infinite ordered sequences in B to be elements of the complement of $\mathcal{U}$.

Using the Nash-Williams type theorem on a directed sets (Theorem 4.7) proved in the previous section, we prove, in Theorem 5.5 below, that every pointwise closed partition family $\mathcal{U}$ of $[X]_*$ is $\mathcal{B}$-completely Ramsey*. Moreover, as we will prove in Theorem 5.7 and Corollary 5.9, every partition family $\mathcal{U}$ of $[X]_*$ which is a Borel subset of $[X]_*$ is $\mathcal{B}$-completely Ramsey*.

Finally, defining the notion of a $\mathcal{B}$-Baire* partition family of $[X]_*$ (in Definition 5.11) we prove in Theorem 5.12 below, that every $\mathcal{B}$-Baire* partition family is $\mathcal{B}$-completely Ramsey*.

Definition 5.1. Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ a coideal basis on X. A subset $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$-**Ramsey*** if for every $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that

$$\textit{either}\ \ [B]_* \subseteq \mathcal{U}\ \ \textit{or}\ \ [B]_* \subseteq [X]_* \setminus \mathcal{U}.$$

If for every $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that $[B]_* \subseteq [X]_* \setminus \mathcal{U}$, then the set $\mathcal{U}$ is called $\mathcal{B}$-**Ramsey* null**.

Definition 5.2. Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ a coideal basis on X. A subset $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$-**completely Ramsey*** if for every $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that

$$\textit{either}\ \ [t,B]_* \subseteq \mathcal{U}\ \ \textit{or}\ \ [t,B]_* \subseteq [X]_* \setminus \mathcal{U}.$$

If for every $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$ there exists $B \in \mathcal{B},\ B \subseteq A$ such that $[t,B]_* \subseteq [X]_* \setminus \mathcal{U}$, then the set $\mathcal{U}$ is called $\mathcal{B}$-**completely Ramsey* null**.

Remarks 5.3. (i) If $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$- completely Ramsey*, then it is $\mathcal{B}$- Ramsey*.
(ii) A set $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$- completely Ramsey* (resp. is $\mathcal{B}$-completely Ramsey* null) if and only if for every $t \in [X]_*^{<\infty}$ the family

$$\mathcal{U}^t = \{\vec{x} - t : \vec{x} \in \mathcal{U} \cap [t,X]_*\}$$

is $\mathcal{B}$- Ramsey* (resp. is $\mathcal{B}$- Ramsey* null), since $[t,B]_* \subseteq \mathcal{U}$, if and only if $[B-t]_* \subseteq \mathcal{U}^t$ (where for $\vec{x} = (x_n)_{n\in\mathbb{N}}$, we set $\vec{x} - t = (x_n)_{n \geq n_0}$ and $n_0 = \min\{n \in \mathbb{N} : t \prec \{x_n\}\}$).
(iii) If $(\mathcal{U}_n)_{n=0}^\infty$ be a sequence of subsets of $[X]_*$, then $(\bigcup_{n=0}^\infty \mathcal{U}_n)^t = \bigcup_{n=0}^\infty \mathcal{U}_n^t$ for every $t \in [X]_*^{<\infty}$.

We endow the set $[X]_*$ with the relative topology of the product topology of $\{0,1\}^X$, which has basic open sets of the form $[t,X]_*$, for $t \in [X]_*^{<\infty}$, where

$$[t,X]_* = \{(x_n)_{n\in\mathbb{N}} \in [X]_* : t \sqsubseteq (x_n)_{n\in\mathbb{N}}\}.$$

Definition 5.4. Let $(X, \prec)$ be an infinite directed set. Identifying every element $\vec{x} = (x_n)_{n\in\mathbb{N}} \in [X]_*$ with the characteristic function $x_{\sigma(\vec{x})} \in \{0,1\}^X$ of the set $\sigma(\vec{x}) = \{x_n : n \in \mathbb{N}\} \subseteq X$, we endow the set $[X]_*$ with the relative topology $\mathfrak{T}$ of the product topology (equivalently by the pointwise convergence topology) of $\{0,1\}^X$. A basis of the topology $\mathfrak{T}$ is the family

$$\{[t, X]_* : \ t \in [X]_*^{<\infty}\}$$

So, a family $\mathcal{U} \subseteq [X]_*$ is **pointwise closed** (equivalently $\mathfrak{T}$-closed) if $\{\chi_{\sigma(\vec{s})} : \vec{s} \in \mathcal{U}\}$ is closed in $\{0,1\}^X$ with the product topology.

Theorem 5.5. *Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$. Then every pointwise closed family $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$-completely Ramsey*.*

Proof. Let $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$. We set

$$\mathcal{F} = \{s \in [X]_*^{<\infty} : \ \text{there exists } \ (z_n)_{n\in\mathbb{N}} \in \mathcal{U} \ \text{ such that } \ s \sqsubseteq (z_n)_{n\in\mathbb{N}}\} \subseteq [X]_*^{<\infty}.$$

According to Theorem 4.7, There exists $B \in \mathcal{B}, \ B \subseteq A$ such that either

(1) $[t, B]_*^{<\infty} \subseteq \mathcal{F}$, or
(2) for every $(x_n)_{n\in\mathbb{N}} \in [t, B]_*$ there exists $s \in [X]_*^{<\infty} \setminus \mathcal{F}$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

If (1) holds, then $[t, B]_* \subseteq \mathcal{U}$. Indeed, let $\vec{x} = (x_n)_{n\in\mathbb{N}} \in [t, B]_*$. Then $(x_1, \dots, x_n) \in [t, B]_*^{<\infty} \subseteq \mathcal{F}$, for every $n \in \mathbb{N}$. Hence, for each $n \in \mathbb{N}$ there exists $\vec{z}_n = (z^n_m)_{m\in\mathbb{N}} \in \mathcal{U}$ such that $(x_1 \dots, x_n) \sqsubseteq (z^n_m)_{m\in\mathbb{N}}$. Since $\mathcal{U}$ is pointwise closed and $(\vec{z}_n)_{n\in\mathbb{N}}$ converges pointwise to $\vec{x}$, we have that $\vec{x} \in \mathcal{U}$. Hence, $[t, B]_* \subseteq \mathcal{U}$.

If (2) holds, then for every $(x_n)_{n\in\mathbb{N}} \in [t, B]_*$ there exists $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$ which belongs to $[X]_*^{<\infty} \setminus \mathcal{F}$. Hence, $[t, B]_* \subseteq [X]_* \setminus \mathcal{U}$. □

Corollary 5.6. *Let $(X, \prec)$ be an infinite directed set and $\mathcal{H} \subseteq [X]^\infty$ a semiselective* coideal on X. Then every pointwise closed family $\mathcal{U} \subseteq [X]_*$ is $\mathcal{H}$-completely Ramsey*.*

Theorem 5.7. *Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$. Then the family of $\mathcal{B}$- completely Ramsey* subsets of $[X]_*$ is closed under the countable unions and also the family of $\mathcal{B}$- completely Ramsey*null subsets of $[X]_*$ is closed under the countable unions.*

Proof. Let $(\mathcal{U}_n)_{n=0}^\infty$ be a sequence of $\mathcal{B}$-completely Ramsey* subsets of $[X]_*^\omega$. We will prove that $\bigcup_{n=0}^\infty \mathcal{U}_n$ is $\mathcal{B}$-completely Ramsey*. In order to prove it, it is enough to prove that $\bigcup_{n=0}^\infty \mathcal{U}_n$ is $\mathcal{B}$-completely Ramsey* null if $\mathcal{U}_n$ is $\mathcal{B}$-completely Ramsey* null for every $n \in \mathbb{N} \cup \{0\}$.

Let $(\mathcal{U}_n)_{n=0}^\infty$ be a sequence of $\mathcal{B}$-completely Ramsey* null subsets of $[X]_*$ and let $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$. We define the families $(\mathcal{R}_s)_{s\in[X]_*^{<\infty}}$ as follows: For $s \in [X]_*^{<\infty}$ such that $t \prec s$ we set

$$\mathcal{R}_s = \{B \in \mathcal{B} : [s, B]_* \cap \ (\mathcal{U}_n)^t = \emptyset \text{ for every } n \in \mathbb{N}, n \leq |s|\},$$

where $|s|$ denotes the cardinality of s and $(\mathcal{U}_n)^t$ is defined in Remarks 5.3. Otherwise, we set $\mathcal{R}_s = \mathcal{B}$.

We will prove that the families $\mathcal{R}_s$, for $s \in [X]_*^{<\infty}$ has the dense-open property. Let $s \in [X]_*^{<\infty}$ such that $t \prec s$ with $|s| = n_0$.

(i) Let $B \in \mathcal{B}$. Since the families $\mathcal{U}_0, \dots \mathcal{U}_{n_0}$ are $\mathcal{B}$-completely Ramsey* null, there exists $C \in \mathcal{B}, \ C \subseteq B$ such that $[t \otimes s, C]_* \cap \mathcal{U}_n = \emptyset$ for every $n \in \{0, \dots, n_0\}$. Hence, $C \in \mathcal{R}_s$.

(ii) Let $B \in \mathcal{R}_s$ and $C \in \mathcal{B}$, $C \subseteq B$. Then, it obviously follows that $C \in \mathcal{R}_s$.

Since $\mathcal{B}$ is a semiselective* coideal basis, there exists $B \in \mathcal{B}, B \subseteq A$ such that for every $s \in [B]_*^{<\infty}$ and $B_1 \subseteq B - s$, $B_1 \in \mathcal{B}$ we have that $B_1 \in \mathcal{R}_s$. Hence, $[\emptyset, B]_* \cap (\mathcal{U}_0)^t = \emptyset$ and consequently $[t, B]_* \cap \mathcal{U}_0 = \emptyset$. In fact $[\emptyset, B]_* \cap (\mathcal{U}_n)^t = \emptyset$ for every $n \in \mathbb{N}$. For a contrary, let $(x_n)_{n\in\mathbb{N}} \in [\emptyset, B]_* \cap (\mathcal{U}_n)^t$ for some $n \in \mathbb{N}$. If $s = (x_1, \ldots, x_n)$, then $B - s \in \mathcal{R}_s$ and consequently $[s, B - s]_* \cap (\mathcal{U}_n)^t = \emptyset$, a contradiction, since $(x_n)_{n\in\mathbb{N}} \in [s, B - s]_* \cap (\mathcal{U}_n)^t$. Thus, $[\emptyset, B]_* \cap (\mathcal{U}_n)^t = \emptyset$ for every $n \in \mathbb{N}$ and consequently $[t, B]_* \cap \mathcal{U}_n = \emptyset$ for every $n \in \mathbb{N}$. This finishes the proof. □

The following Corollaries follow directly from Theorem 5.7.

Corollary 5.8. *Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ be a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$ (or $\mathcal{B} \subseteq [X]^\infty$ be a semiselective* coideal on X). Then the family of all the of $\mathcal{B}$- completely Ramsey* subsets of $[X]_*$ is a σ-field.*

Corollary 5.9. *Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ be a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$(or $\mathcal{B} \subseteq [X]^\infty$ be a semiselective* coideal on X). Then every Borel subset of $[X]_*$ is $\mathcal{B}$-completely Ramsey*.*

Remarks 5.10. (i) The special case of Corollary 5.9 for the coideal $[\mathbb{N}]^\infty$ on $\mathbb{N}$ with the usual order is proved by Galvin-Prikry in [8].

(ii) Corollary 5.9 and Proposition 3.5 imply that every Borel subset of $[[\mathbb{N}]_{>0}^{<\infty}]_*$ is $\mathcal{B}$-completely Ramsey*, where

$$\mathcal{B} = \{FU((F_n)_{n\in\mathbb{N}}) : (F_n)_{n\in\mathbb{N}} \subseteq [\mathbb{N}]_{>0}^{<\infty} \text{ with } F_1 \prec F_2 \prec \ldots\}.$$

This result proved by Milliken, Taylor in [13], [18].

(iii) Every Borel subset of $[L(\Sigma, \vec{k}; \upsilon)]_*$ is $\mathcal{B}$-completely Ramsey*, where

$$\mathcal{B} = \{EV(\vec{w}) : \vec{w} = (w_n)_{n\in\mathbb{N}} \in L^\infty(\Sigma, \vec{k}; \upsilon)\}$$

according to Corollary 5.9 and Proposition 3.4. This result proved by by Farmaki in [4] and earlier by Bergelson-Blass-Hindman in [1] for the particular case of a finite alphabet.

Definition 5.11. Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ be a coideal basis on X. A subset $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$-**Baire*** if for every $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$ there exist $s \in [X]_*^{<\infty}$ and $B \in \mathcal{B}$, $B \subseteq A$ such that $[s, B]_* \subseteq [t, A]_*$ and

$$\text{either } [s, B]_* \subseteq \mathcal{U} \text{ or } [s, B]_* \subseteq [X]_* \setminus \mathcal{U}.$$

If for every $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$ there exist $s \in [X]_*^{<\infty}$ and $B \in \mathcal{B}$ such that $[s, B]_* \subseteq [t, A]_*$ and $[s, B]_* \subseteq [X]_* \setminus \mathcal{U}$, then the set $\mathcal{U}$ is called $\mathcal{B}$-**meager***.

If a subset $\mathcal{U} \subseteq [X]_*$ is $\mathcal{B}$-completely Ramsey*, then it is obviously $\mathcal{B}$-Baire* and of course if $\mathcal{U}$ is $\mathcal{B}$-Ramsey null*, then $\mathcal{U}$ is $\mathcal{B}$-*meager**. The inverse implication does not hold (see Lemma 7.5 in [18]). In the following Theorem we will prove that the two notions coincide if the coideal basis is semiselective*.

Theorem 5.12. *Let $(X, \prec)$ be an infinite directed set and $\mathcal{B} \subseteq [X]^\infty$ a semiselective* coideal basis on X such that $B - s \in \mathcal{B}$ for every $B \in \mathcal{B}$ and $s \in [X]_*^{<\infty}$. Then every $\mathcal{B}$-Baire* subset of $[X]_*$ is completely Ramsey*.*

Proof. Let $\mathcal{U}$ be a $\mathcal{B}$-Baire* subset of $[X]_*$. Let $t \in [X]_*^{<\infty}$ and $A \in \mathcal{B}$. For $s \in [X]_*^{<\infty}$ such that $t \prec s$ we set,

$$\mathcal{R}_s = \{B \in \mathcal{B} : \ [s,B]_* \subseteq \mathcal{U}^t\} \ \cup \ \{B \in \mathcal{B} : [s,B]_* \subseteq [X]_* \setminus \mathcal{U}^t\} \\ \cup \ \{B \in \mathcal{B} : [s,B']_* \nsubseteq \mathcal{U}^t \ \text{ and } \ [s,B']_* \nsubseteq [X]_* \setminus \mathcal{U}^t \ \text{ for every } \ B' \in \mathcal{B}, B' \subseteq B\}.$$

The family $\mathcal{R}_s$ has the dense-open property. Since $\mathcal{B}$ is a semiselective* coideal basis, there exists $B \in \mathcal{B}, B \subseteq A$ such that for every $s \in [B]_*^{<\infty}$ and $B_1 \subseteq B - s,\ B_1 \in \mathcal{B}$ we have that $B_1 \in \mathcal{R}_s$. Then $B - s \in \mathcal{B}$ and $B - s \in \mathcal{R}_s$ for every $s \in [B]_*^{<\infty}$. Let
$\mathcal{F}_1 = \{s \in [B]_*^{<\infty} : [s,B]_* \subseteq \mathcal{U}^t\}$,
$\mathcal{F}_2 = \{s \in [B]_*^{<\infty} : [s,B]_* \subseteq [X]_* \setminus \mathcal{U}^t\}$, and
$\mathcal{F}_3 = \{s \in [B]_*^{<\infty} : [s,B']_* \nsubseteq \mathcal{U}^t$ and $[s,B']_* \nsubseteq [X]_* \setminus \mathcal{U}^t$ for every $B' \in \mathcal{B}, B' \subseteq B\}$.

Since $B - s \in \mathcal{R}_s$ for every $s \in [B]_*^{<\infty}$, we have $[B]_*^{<\infty} = F_1 \cup F_2 \cup F_3$. According to Theorem 4.6, there exists $B_1 \in \mathcal{B},\ B_1 \subseteq B$ such that either

(1) $[B_1]_*^{<\infty} \cap \mathcal{F}_1 = \emptyset$, or
(2) for every $(x_n)_{n\in\mathbb{N}} \in [\emptyset, B_1]_*$ there exists $s \in \mathcal{F}_1$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

If the case (2) holds, then $[\emptyset, B_1]_* \subseteq \mathcal{U}^t$. Otherwise applying Theorem 4.6 for the family $\mathcal{F}_2$ there exists $B_2 \in \mathcal{B},\ B_2 \subseteq B_1$ such that either

(1) $[B_2]_*^{<\infty} \cap \mathcal{F}_2 = \emptyset$, or
(2) for every $(x_n)_{n\in\mathbb{N}} \in [\emptyset, B_2]_*$ there exists $s \in \mathcal{F}_2$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

If the case (2) holds then $[\emptyset, B_2]_* \subseteq [X]_* \setminus \mathcal{U}^t$. Otherwise applying Theorem 4.6 for the family $\mathcal{F}_3$ there exists $B_3 \in \mathcal{B},\ B_3 \subseteq B_2$ such that either

(1) $[B_3]_*^{<\infty} \cap \mathcal{F}_3 = \emptyset$, or
(2) for every $(x_n)_{n\in\mathbb{N}} \in [B_3]_*$ there exists $s \in \mathcal{F}_3$ such that $s \sqsubseteq (x_n)_{n\in\mathbb{N}}$.

We will prove that this last case does not happens.

Since $[B_3]_*^{<\infty} \cap \mathcal{F}_1 = \emptyset$, $[B_3]_*^{<\infty} \cap \mathcal{F}_2 = \emptyset$, we have that $[B_3]_*^{<\infty} \subseteq \mathcal{F}_3$. Since $\mathcal{U}$ has the Baire* property and the family $\mathcal{U}^t$ has this property, so there exists $r \in [X]_*^{<\infty}$ and $B' \in \mathcal{B}, B' \subseteq B_3$ such that $[r,B']_* \subseteq [\emptyset, B_3]_*$ and either $[r,B']_* \subseteq \mathcal{U}^t$ or $[r,B']_* \subseteq [X]_* \setminus \mathcal{U}^t$. Since $r \in [B_3]_*^{<\infty} \subseteq \mathcal{F}_3$ and $B' \subseteq B$ we have that $[r,B']_* \nsubseteq \mathcal{U}^t$ and $[r,B']_* \nsubseteq [X]_* \setminus \mathcal{U}^t$. A contradiction. □

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